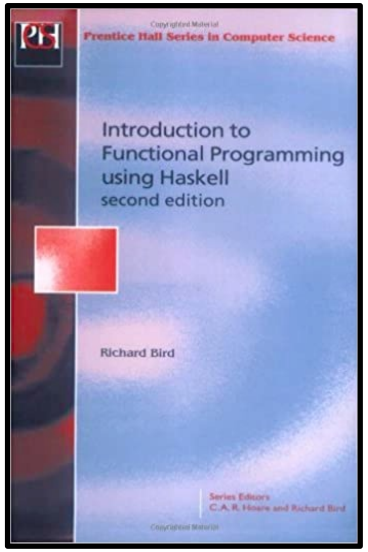
Folding Unfolded

Polyglot FP for Fun and Profit
Haskell and Scala

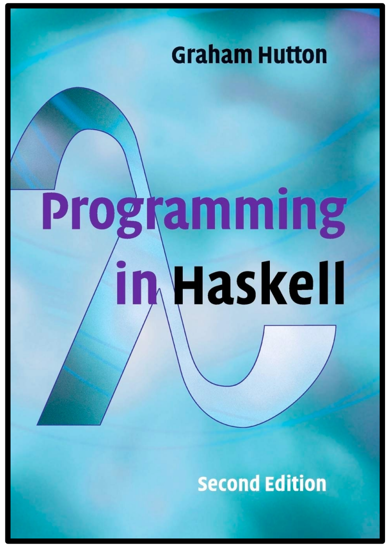


gain a deeper understanding of why **right folds** over very large and **infinite lists** are sometimes possible in **Haskell**

see how **lazy evaluation** and **function strictness** affect **left** and **right folds** in **Haskell**

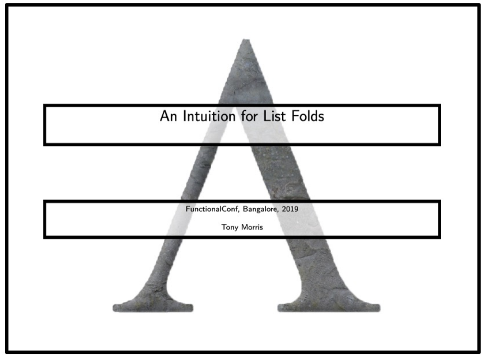
learn when an ordinary **left fold** results in a **space leak** and how to avoid it using a **strict left fold**

Part 5 - through the work of

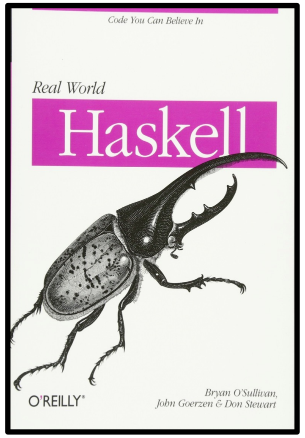


Richard Bird
http://www.cs.ox.ac.uk/people/richard.bird/

Tony Morris
@dibblego



YouTube https://presentations.tmorris.net/



Bryan O'Sullivan
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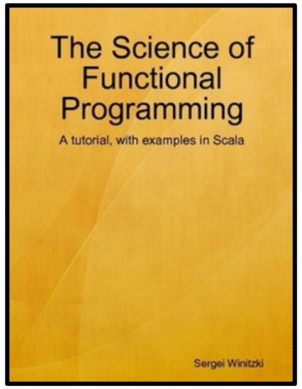

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In Part 4 we said that in this slide deck we were going to cover, in a **Scala** context, the following subjects:

- how to do **right folds** over large lists and **infinite lists**
- how to get around limitations in the applicability of the **accumulator trick**

But I now think that before we do that we ought to get a better understanding of why it is that **right folds** over large lists and **infinite lists** are sometimes possible in **Haskell**. In the process we'll also get a deeper understanding of how **left folds** work: we are in for quite a surprise.

As a result, the original objectives for this slide deck become the objectives for the next deck, i.e. Part 6.

Remember in Part 4 when **Tony Morris** explained the following?

> "whether or not **fold right** will work on an **infinite list** depends on the **strictness** of the function that we are replacing ***Cons*** with"

e.g. if we have an **infinite** list of 1s and we have a **heador** function which when applied to a **default value** and a list returns the value at the head of the list, unless the list is **empty**, in which case it returns ***Nil***, then if we do a **right fold** over the **infinite** list of ones with function **heador** and value 99, we get back 1.

Tony Morris

@dibblego

```
$ heador a = foldr const a
```

OK, so, `heador 99` **`infinity`**.

```
$ heador 99 infinity
```

This will not go all the way to the right of that list, because when it gets there, there is just a ***Cons*** all the way. So I should get 1. And I do:

```
$ heador 99 infinity
1
$
```

i.e. even though the list is **infinite**, rather than **foldr** taking an **infinite** amount of time to evaluate the list, it is able to return a result pretty much instantaneously.

As **Tony** explained, the reason why **foldr** is able to do this is that the **const** function used by **heador** is **lazy**:

> "**`const`**, the function I just used, is **lazy**, it ignores the second argument, and therefore it works on an **infinite list**."

Also, remember in Part 3, when Tony gave this other example of successfully doing a right fold over an infinite list?

Tony Morris

@dibblego

The important thing about fold right to recognize, is that it doesn't do it in any particular order. There is an associativity order, but there is not an execution order. So that is to say, some people might say to me, fold right starts at the right side of the list. This can't be true, because I am going to be passing in an infinite list, which doesn't have a right side, and I am going to get an answer. If it started at the right, it went a really long way, and it is still going. So that is what I should see if that statement is true, but I don't see that. It associates to the right, it didn't start executing from the right. It's a subtle difference.

What if I have a list of booleans and I want to and them all up? What am I going to replace *Nil* with? Not 99. True. Yes.

and (&&) the booleans of a list

Supposing

```
list = Cons True (Cons True (Cons False (Cons True Nil)))
```

So if I have the above list, and I replace *Nil* with True and *Cons* with (&&), like this

and (&&) the booleans of a list

- let *Cons* = (&&)
- let *Nil* = True

Supposing

```
list = (&&) True ((&&) True ((&&) False ((&&) True True)))

conjunct list = foldr (&&) True list
conjunct = foldr (&&) True
```

It will and (&&) them all up

So there is the code. Right fold replacing *Cons* with (&&) and *Nil* with True. It doesn't do it in any order. I could have an infinite list of booleans. Suppose I had an infinite list of booleans and it started at False. *Cons* False something. And I said foldr (&&) True. I should get back False. And I do. So clearly it didn't start from the right. It never went there. It just saw the False and stopped.

@philip_schwarz

Although **Tony** didn’t say it explicitly, the reason why **foldr** works in the example on the previous slide is that && is **non-strict** in its second parameter whenever its first parameter is **False**.

Before we look at what it means for a function to be **strict**, we need to understand what **lazy evaluation** is, so in the next 10 slides we are going to see how **Richard Bird** and **Graham Hutton** explain this concept. If you are already familiar with **lazy evaluation** then feel free to skip the slides.

1.2 Evaluation

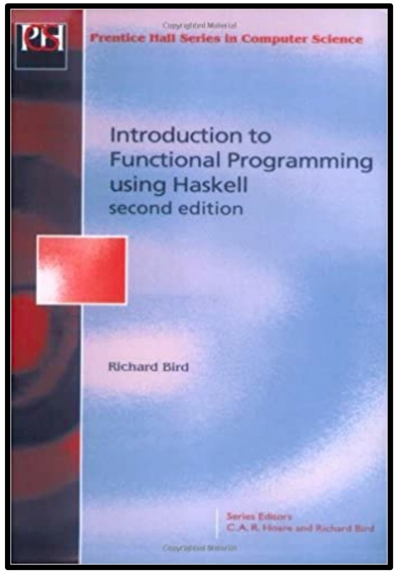
The computer evaluates an expression by reducing it to its simplest equivalent form and displaying the result. The terms ***evaluation***, ***simplification*** and ***reduction*** will be used interchangeably to describe this process. To give a brief flavour, consider the expression $square\ (3+4)$; one possible sequence is

$$
\begin{aligned}
& square\ (3+4) \\
= \quad & \{\text{definition of } +\} \\
& square\ 7 \\
= \quad & \{\text{definition of } square\} \\
& 7 \times 7 \\
= \quad & \{\text{definition of } \times\} \\
& 49
\end{aligned}
$$

The first and third step refer to the use of the built-in rules for addition and multiplication, while the second step refers to the use of the rule defining $square$ supplied by the programmer. That is to say, the definition of $square\ x = x \times x$ is interpreted by the computer simply as a **left-to-right rewrite rule** for reducing expressions involving $square$. The expression '49' cannot be further reduced, so that is the result displayed by the computer. An expression is said to be ***canonical***, or in ***normal form***, if it cannot be further reduced. Hence '49' is in **normal form**.

Another reduction sequence for $square\ (3+4)$ is

$$
\begin{aligned}
& square\ (3+4) \\
= \quad & \{\text{definition of } square\}
\end{aligned}
$$



Richard Bird

$(3+4) \times (3+4)$

= $\{$ definition of $+\}$

$7 \times (3+4)$

= $\{$ definition of $+\}$

7×7

= $\{$ definition of $\times\}$

49

In this **reduction sequence** the rule for *square* is applied first, but the final result is the same. **A characteristic feature of functional programming is that if two different reduction sequences both terminate then they lead to the same result**. In other words, **the meaning of an expression is its value and the task of the computer is simply to obtain it**. Let us give another example. Consider the script

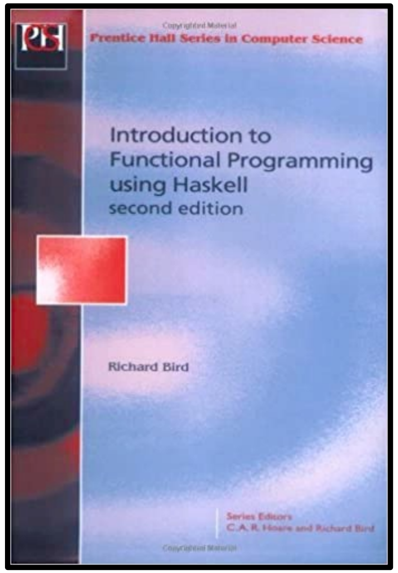
$$\begin{aligned} &three :: Integer \rightarrow Integer \\ &three\ x = 3 \end{aligned}$$

$$\begin{aligned} &infinity :: Integer \\ &infinity = infinity + 1 \end{aligned}$$

It is not clear what integer, if any, is defined by the second equation, but the computer can nevertheless use the equation as a rewrite rule. Now consider **simplification** of *three infinity*. If we try to **simplify** *infinity* first, then we get the **reduction sequence**

three infinity

= $\{$ definition of $infinity\}$



Richard Bird

$three\,(infinity + 1)$

= { definition of $infinity$ }

$three\,((infinity + 1) + 1)$

= { and so on … }

…

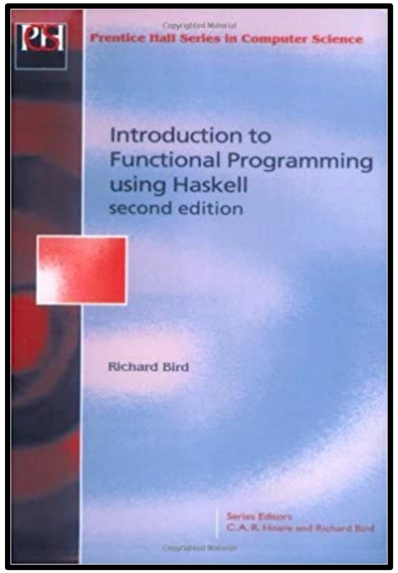
This reduction sequence does not terminate. If on the other hand we try to simplify $three$ first, then we get the sequence

$three\ infinity$

= { definition of $three$ }

3

This sequence terminates in one step. So some ways of simplifying an expression may terminate while others do not. In Chapter 7 we will describe a reduction strategy, called *lazy evaluation*, that guarantees termination whenever termination is possible, and is also reasonably efficient. Haskell is a lazy functional language, and we will explore what consequences such a strategy has in the rest of the book. However, whichever strategy is in force, the essential point is that expressions are evaluated by a conceptually simple process of substitution and simplification, using both primitive rules and rules supplied by the programmer in the form of definitions.



Richard Bird

7.1 Lazy Evaluation

Let us start by revisiting the evaluation of $square\ (3+4)$ considered in Chapter 1.

Recall that one **reduction sequence** is

$$\begin{array}{ll} & square\ (3+4) \\ = & \quad \{\text{ definition of } + \} \\ & square\ 7 \\ = & \quad \{\text{ definition of } square \} \\ & 7 \times 7 \\ = & \quad \{\text{ definition of } \times \} \\ & 49 \end{array}$$

and another **reduction sequence** is

$$\begin{array}{ll} & square\ (3+4) \\ = & \quad \{\text{ definition of } square \} \\ & (3+4) \times (3+4) \\ = & \quad \{\text{ definition of } + \} \\ & 7 \times (3+4) \\ = & \quad \{\text{ definition of } + \} \\ & 7 \times 7 \\ = & \quad \{\text{ definition of } \times \} \\ & 49 \end{array}$$

These two **reduction sequences** illustrate two **reduction policies**, called ***innermost*** and ***outermost*** **reduction**, respectively. In the first **sequence**, each step **reduces** an **innermost redex**. The word '**redex**' is short for '**reducible expression**', and an **innermost redex** is one that contains no other **redex**. In the second **sequence**, each step reduces an **outermost redex**. An **outermost redex** is one that is contained in no other **redex**.

Here is another example. First, **innermost reduction**:

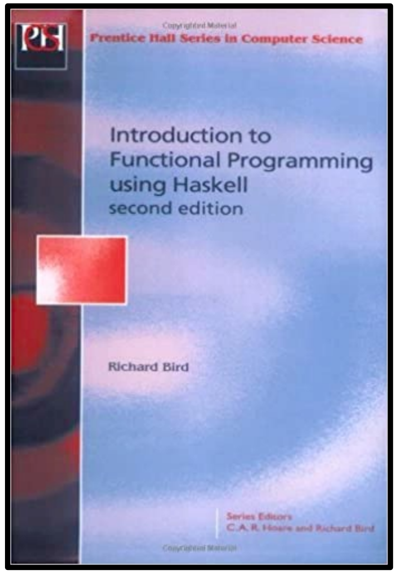
$$\begin{array}{ll} & fst\ (square\ 4,\ square\ 2) \\ = & \quad \{\text{ definition of } square \} \\ & fst\ (4 \times 4,\ square\ 2) \\ = & \quad \{\text{ definition of } \times \} \\ & fst\ (16,\ square\ 2) \\ = & \quad \{\text{ definition of } square \} \\ & fst\ (16,\ 2 \times 2) \\ = & \quad \{\text{ definition of } \times \} \\ & fst\ (16,\ 4) \\ = & \quad \{\text{ definition of } fst \} \\ & 16 \end{array}$$

The **innermost reduction** takes five steps. In the first two steps there was a choice of **innermost redexes** and the leftmost **redex** is chosen.

The **outermost reduction** policy for the same expression yields

$$\begin{array}{ll} & fst\ (square\ 4,\ square\ 2) \\ = & \quad \{\text{ definition of } fst \} \\ & square\ 4 \\ = & \quad \{\text{ definition of } square \} \\ & 4 \times 4 \\ = & \quad \{\text{ definition of } \times \} \\ & 16 \end{array}$$

The **outermost reduction** sequence takes three steps. By using **outermost reduction**, evaluation of $square\ 2$ was avoided.



Richard Bird

The two **reduction policies** have different characteristics. Sometimes **outermost reduction** will give an answer when **innermost reduction** fails to **terminate** (consider replacing $square\ 2$ by $undefined$ in the expression above). However, if both methods **terminate**, then they give the same result.

Outermost reduction has the important property that if an expression has a **normal form**, then **outermost reduction** will compute it. **Outermost reduction** is also called **normal-order** on account of this property. It would seem therefore, that **outermost reduction** is a better choice than **innermost reduction**, but there is a catch. As the first example shows, **outermost reduction** can sometimes require more steps than **innermost reduction**. The problem arises with any function whose definition contains repeated occurrences of an argument. By binding such an argument to a suitably large expression, the difference between **innermost** and **outermost reduction** can be made arbitrarily large.

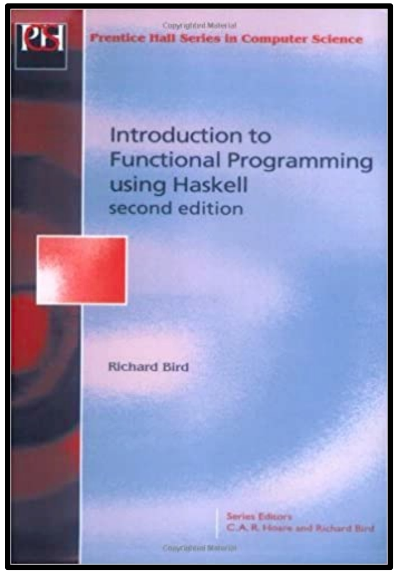
This problem can be solved by representing expressions as *graphs* rather than trees. Unlike trees, graphs can share subexpressions. For example, the graph

$$(\bullet \times \bullet) \qquad (3+4)$$

represents the expression $(3+4) \times (3+4)$. Each occurrence of $(3+4)$ is represented by an arrow, called a *pointer*, to a single instance of $(3+4)$. Now using **outermost *graph* reduction** we have

$$\begin{aligned} & square\ (3+4) \\ = & \quad \{\text{ definition of } square \text{ }\} \\ & (\bullet \times \bullet) \qquad (3+4) \\ = & \quad \{\text{ definition of } + \text{ }\} \\ & (\bullet \times \bullet) \qquad 7 \\ = & \quad \{\text{ definition of } \times \text{ }\} \\ & 49 \end{aligned}$$

The reduction has only three steps. The representation of expressions as graphs means that duplicated subexpressions can be shared and **reduced** at most once. With **graph reduction**, **outermost reduction** never takes more steps than **innermost reduction**. Henceforth, we will refer to **outermost graph reduction** by its common name, **lazy evaluation**, and to **innermost graph reduction** as **eager evaluation**.



Richard Bird

15.2 Evaluation Strategies

…

When evaluating an expression, in what order should the **reductions** be performed? One common strategy, known as **innermost evaluation**, is to always choose a **redex** that is **innermost**, in the sense that it contains no other **redex**. If there is more than one innermost **redex**, by convention we choose the one that begins at the leftmost position in the expression.

…

Innermost evaluation can also be characterized in terms of how arguments are passed to functions. In particular, using this strategy ensures that **arguments are always fully evaluated before functions are applied**. **That is, arguments are passed by value**. For example, as shown above, evaluating `mult (1+2,2+3)` using **innermost evaluation** proceeds by first evaluating the arguments 1+2 and 2+3, and then applying `mult`. **The fact that we always choose the leftmost innermost redex ensures that the first argument is evaluated before the second**.

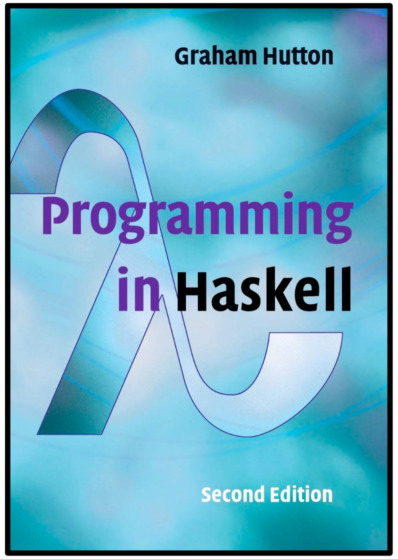
…

In terms of how arguments are passed to functions, using **outermost evaluation** allows functions to be applied before their arguments are evaluated. **For this reason, we say that arguments are passed by name**. For example, as shown above, evaluating `mult(1+2,2+3)` using **outermost evaluation** proceeds by first applying the function `mult` to the two unevaluated arguments 1+2 and 2+3, and then evaluating these two expressions in turn.

Lambda expressions

…

Note that in **Haskell**, the selection of **redexes** within the bodies of **lambda expressions** is prohibited. The rationale for not '**reducing** under lambdas' is that functions are viewed as black boxes that we are not permitted to look inside. More formally, the only operation that can be performed on a function is that of applying it to an argument. As such, reduction within the body of a function is only permitted once the function has been applied. For example, the function \x → 1 + 2 is deemed to already be fully evaluated, even though its body contains the **redex** 1 + 2, but once this function has been applied to an argument, evaluation of this



Graham Hutton
@haskellhutt

redex can then proceed:

$$\begin{array}{ll} & (\backslash x \to 1 + 2)\ 0 \\ = & \quad \{\text{ applying the lambda }\} \\ & 1 + 2 \\ = & \quad \{\text{ applying } + \} \\ & 3 \end{array}$$

Using innermost and outermost evaluation, but not within lambda expressions, is normally referred to as call-by-value and call-by-name evaluation, respectively. In the next two sections we explore how these two evaluation strategies compare in terms of two important properties, namely their termination behaviour and the number of reduction steps that they require.

15. 3 Termination

…

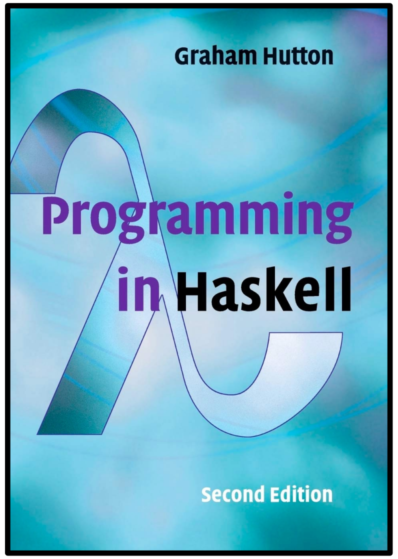
call-by-name evaluation may produce a result when call-by-value evaluation fails to terminate. More generally, we have the following important property: if there exists any evaluation sequence that terminates for a given expression, then call-by-name evaluation will also terminate for this expression, and produce the same final result. In summary, call-by-name evaluation is preferable to call-by-value for the purpose of ensuring that evaluation terminates as often as possible.

…

15.4 Number of reductions

…

call-by-name evaluation may require more reduction steps than call-by-value evaluation, in particular when an argument is used more than once in the body of a function. More generally, we have the following property: arguments are evaluated precisely once using call-by-value evaluation, but may be evaluated many times using call-by-name. Fortunately, the above efficiency problem with call-by-name evaluation can easily be solved, by using pointers to indicate sharing of expressions during evaluation. That is, rather than physically copying an argument if it is used many times in the body of a function, we simply keep one copy of the argument and make many pointers to it. In this manner, any reductions that are performed on the argument are automatically shared between each of the pointers to that argument. For example, using this strategy we have:



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$square\ (1 + 2)$

= $\{$ applying $square$ $\}$

$(\bullet * \bullet)$ $\quad (1 + 2)$

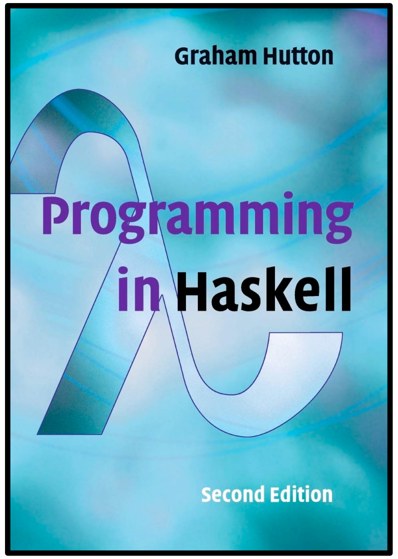
= $\{$ applying $+$ $\}$

$(\bullet * \bullet)$ $\quad 3$

= $\{$ applying $*$ $\}$

3

That is, when applying the definition $square\ n = n * n$ in the first step, we keep a single copy of the argument expression 1+2, and make two pointers to it. In this manner, when the expression 1+2 is **reduced** in the second step, both pointers in the expression share the result. **The use of call-by-name evaluation in conjunction with sharing is known as lazy evaluation**. **This is the evaluation strategy that is used in Haskell**, **as a result of which Haskell is known as a lazy programming language**. **Being based upon call-by-name evaluation**, **lazy evaluation has the property that it ensures that evaluation terminates as often as possible**. Moreover, **using sharing ensures that lazy evaluation never requires more steps than call-by-value evaluation**. The use of the term '**lazy**' will be explained in the next section.



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15.5 Infinite structures

An additional property of call-by-name evaluation, and hence lazy evaluation, is that it allows what at first may seem impossible: programming with infinite structures. We have already seen a simple example of this idea earlier in this chapter, in the form of the evaluation of `fst (0,inf)` avoiding the production of the infinite structure `1 + (1 + (1 + ...))` defined by inf. More interesting forms of behaviour occur when we consider infinite lists. For example, consider the following recursive definition:

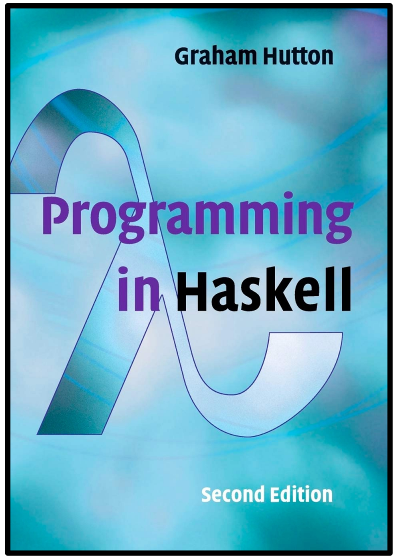
$$\begin{array}{l} ones :: [Int] \\ ones = 1 : ones \end{array}$$

That is, the list *ones* is defined as a single one followed by itself. As with `inf`, evaluating *ones* does not terminate, regardless of the strategy used:

$$\begin{array}{ll} & ones \\ = & \quad \{ \text{applying } ones \} \\ & 1 : ones \\ = & \quad \{ \text{applying } ones \} \\ & 1 : (1 : ones) \\ = & \quad \{ \text{applying } ones \} \\ & 1 : (1 : (1 : ones)) \\ = & \quad \{ \text{applying } ones \} \\ & \ldots \end{array}$$

In practice, evaluating *ones* using GHCi will produce a never-ending list of ones,

$> ones$
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, …



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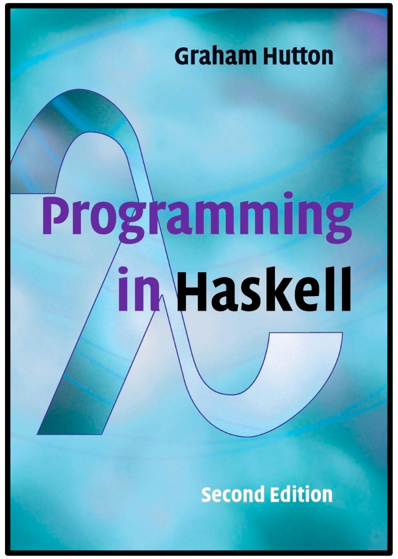
Now consider the expression *head ones*, where *head* is the library function that selects the first element of a list, defined by $head\,(x : \_\,) = x$. Using **call-by-value** evaluation in this case also results in **non-termination**.

$$
\begin{aligned}
& head\ ones \\
= \quad & \{\text{ applying } ones \text{ }\} \\
& head\ (1 : ones) \\
= \quad & \{\text{ applying } ones \text{ }\} \\
& head\ (1 : (\ 1 : ones)) \\
= \quad & \{\text{ applying } ones \text{ }\} \\
& head\ (1 : (\ 1 : (\ 1 : ones))) \\
= \quad & \{\text{ applying } ones \text{ }\} \\
& \ldots
\end{aligned}
$$

In contrast, using **lazy evaluation** (or **call-by-name evaluation**, as sharing is not required in this example), results in **termination** in two steps:

$$
\begin{aligned}
& head\ ones \\
= \quad & \{\text{ applying } ones \text{ }\} \\
& head\ (1 : ones) \\
= \quad & \{\text{ applying } head \text{ }\} \\
& 1
\end{aligned}
$$

This behaviour arises because lazy evaluation proceeds in a lazy manner as its name suggests, only evaluating arguments as and when this is strictly necessary in order to produce results. For example, when selecting the first element of a list, the remainder of the list is not required, and hence in $head\ (1 : ones)$ the further evaluation of the infinite list *ones* is avoided. More generally, **we have the following property: using lazy evaluation, expressions are only evaluated as much as required by the context in which they are used**. Using this idea, we now see that **under lazy evaluation** *ones* **is not an infinite list as such, but rather a potentially infinite list, which is only evaluated as much as required by the context**. This idea is not restricted to lists, but applies equally to any form of data structure in **Haskell**.



Graham Hutton
@haskellhutt

After that introduction to **lazy evaluation**, we can now look at what it means for a function to be **strict**.

In the next two slides we see how **Richard Bird** explains the concept.

1.3 Values

The evaluator for a **functional language** prints a value by printing its **canonical representation**; this **representation** is dependent both on the syntax given for forming expressions, and the precise definition of the **reduction** rules.

Some values have no **canonical representations**, for example function values. …
…

Other values may have reasonable **representations**, but no finite ones. For example, the number π …
…

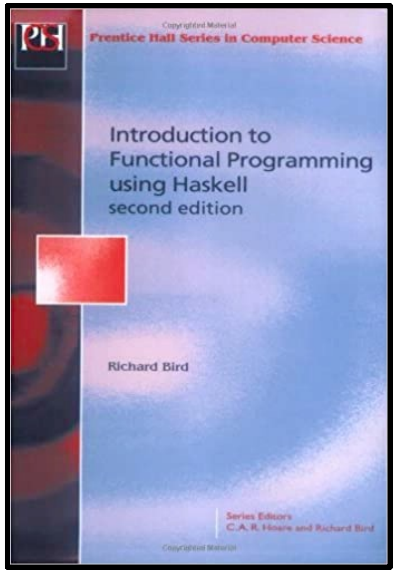
For some expressions the process of **reduction** never stops and never produces any result. For example, the expression *infinity* defined in the previous section leads to an **infinite reduction sequence**. Recall that the definition was

$$\begin{aligned} \mathit{infinity} &:: \mathit{Integer} \\ \mathit{infinity} &= \mathit{infinity} + 1 \end{aligned}$$

Such expressions do not denote **well-defined values** in the normal mathematical sense. As another example, assuming the operator / denotes numerical division, returning a number of type `Float`, the expression 1/0 does not denote a well-defined floating point number. A request to evaluate 1/0 may cause the evaluator to respond with an error message, such as ‘attempt to divide by zero’, or go into an **infinitely long** sequence of calculations without producing any result.

In order that we can say that, without exception, every syntactically well-formed expression denotes a value, it is convenient to introduce a special symbol $\perp$, pronounced ’bottom’, to stand for the undefined value of a particular type. In particular, the value of *infinity* is the undefined value $\perp$ of type `Integer`, and 1/0 is the undefined value $\perp$ of type `Float`. Hence we can assert that $1/0 = \perp$.

The computer is not expected to be able to produce the value $\perp$. Confronted with an expression whose value is $\perp$, the computer may give an error message or it may remain perpetually silent. The former situation is detectable, but the second one is not (after all, evaluation might have **terminated** normally the moment the programmer decided to abort it). Thus $\perp$ is a special kind of value, rather like the special value ∞ in mathematical calculus. Like special values in other branches of mathematics, $\perp$ can be admitted to




Richard Bird

the universe of values oly if we state precisely the properties it is required to have and its relationship with other values.

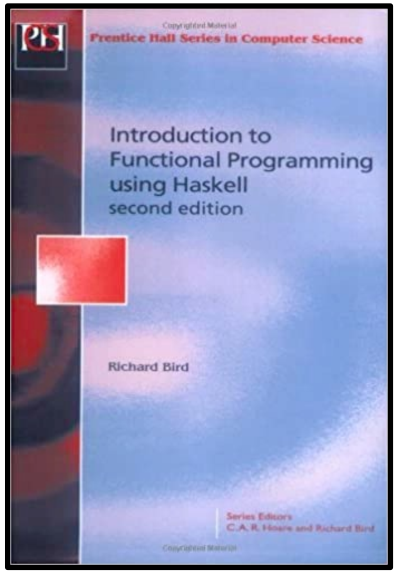
It is possible, conceptually at least, to apply functions to $\perp$. For example, with the definitions $three\ x = 3$ and $square\ x = x \times x$, we have

? *three infinity*
3

? *square infinity*
{ Interrupted! }

In the first evaluation the value of *infinity* was not needed to compute the calculation, so it was never calculated. This is a consequence of the **lazy evaluation reduction strategy** mentioned earlier. On the other hand, in the second evaluation the value of *infinity* is needed to complete the computation: one cannot compute $x \times x$ without knowing the value of x. Consequently, the evaluator goes into an **infinite reduction sequence** in an attempt to simplify *infinity* to **normal form**. Bored by waiting for an answer that we know will never come, we hit the interrupt key.

If $f\ \perp = \perp$, then f is said to be a strict function; otherwise it is nonstrict. Thus, *square* is a strict function, while *three* is nonstrict. Lazy evaluation allows nonstrict functions to be defined, some other strategies do not.



Richard Bird

And here is how **Richard Bird** describes the fact that **&&**, which he calls $\wedge$, is **strict**.

Two basic functions on **Booleans** are the operations of **conjunction**, denoted by the binary operator $\wedge$, and **disjunction**, denoted by $\vee$. These operations can be defined by

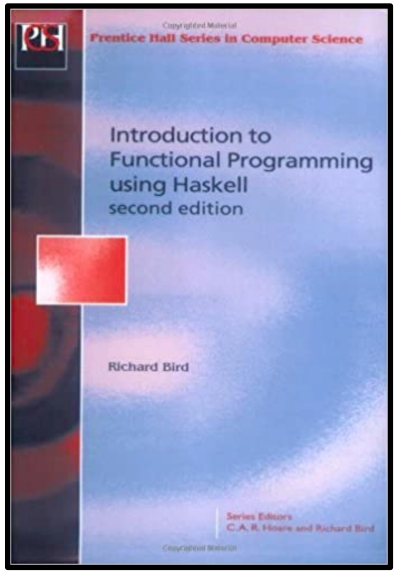
$$
\begin{aligned}
&(\wedge), (\vee) \quad :: \mathit{Bool} \rightarrow \mathit{Bool} \rightarrow \mathit{Bool}\\
&\mathit{False} \wedge x = \mathit{False}\\
&\mathit{True} \wedge x = x
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{False} \vee x = x\\
&\mathit{True} \vee x = \mathit{True}
\end{aligned}
$$

The definitions use pattern matching on the left-hand argument. For example, in order to simplify expressions of the form $e_1 \wedge e_2$, the computer first reduces e_1 to normal form. If the result is *False* then the first equation for $\wedge$ is used, so the computer immediately returns *False*. If e_1 reduces to *True*, then the second equation is used, so e_2 is evaluated. It follows from this description of how pattern matching works that

$$
\begin{aligned}
&\perp \wedge \mathit{False} = \perp\\
&\mathit{False} \wedge \perp = \mathit{False}\\
&\mathit{True} \wedge \perp = \perp
\end{aligned}
$$

Thus $\wedge$ is strict in its left-hand argument, but not strict in its right-hand argument. Analogous remarks apply to $\vee$.



Richard Bird

Now that we have a good understanding of the concepts of **lazy evaluation** and **strictness**, we can revisit the two examples in which **Tony Morris** showed that if a binary function is **nonstrict** in its second argument then it is sometimes possible to successfully do a **right fold** of the function over an **infinite** list.

```
λ> infinity = 1 : infinity
λ> const x y = x
λ> heador a = foldr const a
```

```
$ infinity
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 etc, etc
```

$$
\begin{aligned}
& heador\ 99\ \ infinity \\
= & \quad \{ \text{definition of } heador \} \\
& foldr\ const\ 99\ \ infinity \\
= & \quad \{ \text{definition of } infinity \} \\
& foldr\ \ const\ \ 99\ (1 : infinity\ ) \\
= & \quad \{ \text{definition of } foldr \} \\
& const\ \ 1\ (foldr\ const\ 99\ infinity\ ) \\
= & \quad \{ \text{definition of } const \} \\
& 1
\end{aligned}
$$

```
λ> heador 99 infinity
1
```

$$
\begin{aligned}
& foldr && :: (\alpha \rightarrow \beta \rightarrow \beta) \rightarrow \beta \rightarrow [\alpha] \rightarrow \beta \\
& foldr\ f\ e\ [\,] && = e \\
& foldr\ f\ e\ (x : xs) && = f\ x\ (foldr\ f\ e\ xs)
\end{aligned}
$$

$$
\begin{aligned}
& infinity\ :: [Integer] \\
& infinity = 1 : infinity
\end{aligned}
$$

$$
\begin{aligned}
& const && :: \alpha \rightarrow \beta \rightarrow \alpha \\
& const\ x\ y && = x
\end{aligned}
$$

```
λ> const 99 undefined
99
```

$$
\begin{aligned}
& heador && :: \alpha \rightarrow [\alpha] \rightarrow \alpha \\
& heador\ a && = foldr\ const\ a
\end{aligned}
$$

```
λ> heador 99 (1 : undefined)
1
```

Lazy evaluation causes just enough of *infinity* to be valuated to allow *foldr* to be invoked.

@philip_schwarz

Neither *const* nor *heador* are **strict** in their second argument.

```
λ> :{
(&&) False x = False
(&&) True x = x
:}
λ> infinity = False : infinity
λ> conjunct = foldr (&&) True
```

```
λ> infinity
[False,False,False,False,False,False,False,False
,False,False,False,False,False,False, etc, etc
```

Lazy evaluation causes just enough of $infinity$ to be valuated to allow $foldr$ to be invoked.

$conjunct\ infinity$

= { definition of $conjunct$ }

$foldr\ (\&\&)\ True\ infinity$

= { definition of $infinity$ }

$foldr\ (\&\&)\ True\ (False : infinity)$

= { definition of $foldr$ }

$False\ \&\&\ (foldr\ (\&\&)\ True\ infinity)$

= { definition of && }

$False$

```
λ> conjunct infinity
False
```

$$
\begin{aligned}
&foldr && :: (\alpha \to \beta \to \beta) \to \beta \to [\alpha] \to \beta \\
&foldr\ f\ e\ [\,] && = e \\
&foldr\ f\ e\ (x{:}xs) && = f\ x\ (foldr\ f\ e\ xs)
\end{aligned}
$$

$$
\begin{aligned}
&(\&\&) && :: Bool \to Bool \to Bool \\
&False\ \&\&\ x && = False \\
&True\ \&\&\ x && = x
\end{aligned}
$$

```
λ> False && undefined
False
```

$$
\begin{aligned}
&infinity :: [Bool] \\
&infinity = False : infinity
\end{aligned}
$$

$$
\begin{aligned}
&conjunct && :: [Bool] \to Bool \\
&conjunct && = foldr\ (\&\&)\ True
\end{aligned}
$$

```
λ> conjunct (False:undefined)
False
```

(&&) is **strict** in its second argument only when its first argument is **True**.

```
λ> infinity = True : infinity
λ> conjunct infinity
{ Interrupted }
```

If we **rigth fold** (&&) over a list then the **folding** ends as soon as a `False` is encountered. e.g. if the first element of the list is `False` then the **folding** ends immediately. If the list we are **folding** over is **infinite**, then if no `False` is encountered the **folding** never ends. Note that because of how (&&) works, there is no need to keep building a growing **intermediate expression** during the **fold**: memory usage is constant.

$$\begin{aligned} (\&\&) &:: Bool \rightarrow Bool \rightarrow Bool \\ False\ \&\&\ x &= False \\ True\ \&\&\ x &= x \end{aligned}$$

Right Fold Scenario	**Code**	**Result**	**Approximate Duration**
List Size: **huge** Function Strictness: nonstrict in 2nd argument when 1st is False	`foldr (&&) True (replicate 1,000,000,000 True)`	True	38 seconds Ghc memory: • initial: 75.3 MB • final: 75.3 MB
List Size: **huge** Function Strictness: nonstrict in 2nd argument when 1st is False	`foldr (&&) True (replicate 1,000,000,000 False)`	False	0 seconds • initial: 75.3 MB • final: 75.3 MB
List Size: **infinite** Function Strictness: nonstrict in 2nd argument when 1st is False	`trues = True : trues` `foldr (&&) True trues`	Does not terminate Keeps going	I stopped it after 3 min. • initial: 75.3 MB • final: 75.3 MB
List Size: **infinite** Function Strictness: nonstrict in 2nd argument when 1st is False	`falses = False : falses` `foldr (&&) True falses`	False	0 Seconds

Let’s contrast that with what happens when the function with which we are doing a **right fold** is **strict** in both of its arguments.

e.g. we are able to successfully **right fold** (+) over a large list, but if the list is huge or outright **infinite**, then **folding** fails with a **stack overflow** exception, because the growing **intermediate expression** that gets built, and which represents the sum of all the list’s elements, eventually exhausts the available **stack memory**.

Right Fold Scenario (continued)	Code	Result	Approximate Duration
List Size: large Function Strictness: strict in **both** arguments	`foldr (+) 0 [1..10,000,000]`	50000005000000	2 seconds
List Size: huge Function Strictness: strict in **both** arguments	`foldr (+) 0 [1..100,000,000]`	**** Exception: stack overflow*	3 seconds
List Size: infinite Function Strictness: strict in **both** arguments	`foldr (+) 0 [1..]`	**** Exception: stack overflow*	3 seconds

@philip_schwarz

As we know, the reason why on the previous slide we saw **foldr** encounter a **stack overflow** exception when processing an **infinite** list, or a sufficiently long list, is that **foldr** is not **tail recursive**.

So just for completeness, let's go through the same scenarios as on the previous slide, but this time using **foldl** rather than **foldr**. Since **foldl** behaves like a loop, it should not encounter any **stack overflow** exception due to processing an **infinite** list or a sufficiently long list.

Left Fold Scenario	Code	Result	Approximate Duration
List Size: **large** Function Strictness: strict in **both** arguments	`foldl (+) 0 [1..10,000,000]`	50000005000000	4 seconds Ghc memory: • initial: 27.3MB • final: 1.10 GB
List Size: **huge** Function Strictness: strict in **both** arguments	`foldl (+) 0 [1..100,000,000]`	**** Exception: stack overflow*	3 seconds Ghc memory: • initial: 27.3MB • final: 10 GB
List Size: **infinite** Function Strictness: strict in **both** arguments	`foldl (+) 0 [1..]`	Does not terminate Keeps going	I stopped it after 3 min Ghc memory: • initial: 27.3MB • final: 22 GB

That was a bit of a surprise!

When the list is **infinite**, **foldl** does not **terminate**, which is what we expect, given that a **left fold** is like a **loop**.

Tony Morris
@dibblego

the key intuition
- left fold performs a *loop*, just like we are familiar with
- right fold performs *constructor replacement*

The key intuition is, the thing to take aways is, a **left fold** does a **loop**, and a **right fold** does **constructor replacement**.

If you always remember those two things you'll never go wrong.

from this we derive some observations
- left fold will *never* work on an infinite list
- right fold *may* work on an infinite list

Left fold will never work on an **infinite list**. We can see that in the **loop**. **Right fold** might. And these are just independent observations. They have nothing to do with programming languages. I have used **Haskell** as the example. These things are independent of the programming language.

But surprisingly, when the list is **finite** yet sufficiently large, **foldl** encounters a **stack overflow** exception!

How can that be? Shouldn't the fact that **foldl** is **tail recursive** guarantee that it is **stack safe**? If you need a refresher on **tail recursion** then see the next three slides, otherwise you can just skip them.

Also, note how the larger the list, the more **heap** space **foldl** uses. Why is **foldl** using so much **heap** space? Isn't it supposed to only require space for an **accumulator** that holds an **intermediate result**? Again, if you need a refresher on **accumulators** and **intermediate results** then see the next three slides.

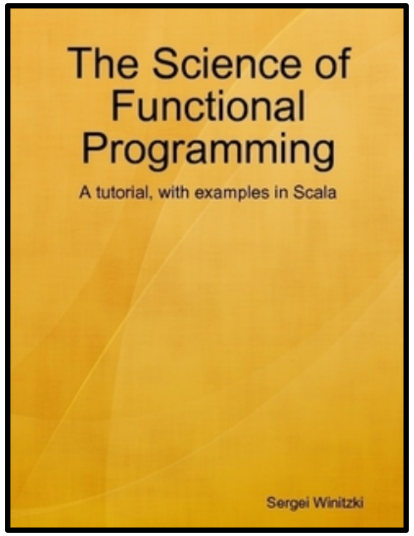
2.2.3 Tail recursion

The code of lengthS will fail for large enough sequences. To see why, consider an inductive definition of the `.length` method as a function `lengthS`:

```
def lengthS(s: Seq[Int]): Int =
  if (s.isEmpty) 0
  else 1 + lengthS(s.tail)

scala> lengthS((1 to 1000).toList)
  res0: Int = 1000

scala> val s = (1 to 100_000).toList
s : List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22,
23, 24, 25, 26, 27, 28, 29, 30, 31, 32, ...

scala> lengthS(s)
java.lang.StackOverflowError
at .lengthS(<console>:12)
at .lengthS(<console>:12)
at .lengthS(<console>:12)
at .lengthS(<console>:12)
...
```

The problem is not due to insufficient main memory: we are able to compute and hold in memory the entire sequence `s`. The problem is with the code of the function `lengthS`. This function calls itself inside the expression `1 + lengthS(...)`. So we can visualize how the computer evaluates this code:

```
lengthS(Seq(1, 2, ..., 100000))
= 1 + lengthS(Seq(2, ..., 100000))
= 1 + (1 + lengthS(Seq(3, ..., 100000)))
= ...
```



Sergei Winitzki

sergei-winitzki-11a6431

```scala
def lengthS(s: Seq[Int]): Int =
  if (s.isEmpty) 0
  else 1 + lengthS(s.tail)
```

```scala
lengthS(Seq(1, 2, ..., 100000))
= 1 + lengthS(Seq(2, ..., 100000))
= 1 + (1 + lengthS(Seq(3, ..., 100000)))
= ...
```

The function body of `lengthS` will evaluate the inductive step, that is, the "else" part of the "if/else", about 100_000 times. Each time, the sub-expression with nested computations `1+(1+(...))` will get larger.

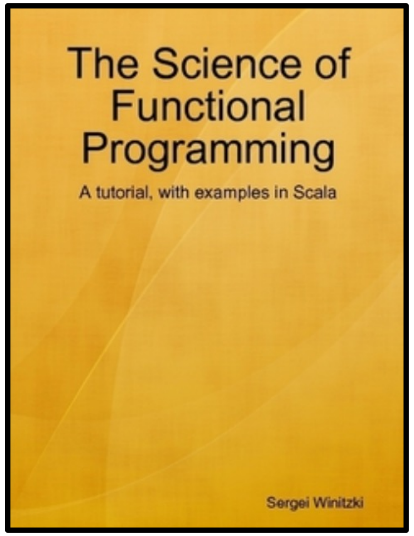
This intermediate sub-expression needs to be held somewhere in memory, until at some point the function body goes into the base case and returns a value. When that happens, the entire intermediate sub-expression will contain about 100_000 nested function calls still waiting to be evaluated.

This sub-expression is held in a special area of memory called stack memory, where the not-yet-evaluated nested function calls are held in the order of their calls, as if on a "stack". Due to the way computer memory is managed, the stack memory has a fixed size and cannot grow automatically. So, when the intermediate expression becomes large enough, it causes an overflow of the stack memory and crashes the program.

A way to avoid stack overflows is to use a trick called tail recursion. Using tail recursion means rewriting the code so that all recursive calls occur at the end positions (at the "tails") of the function body. In other words, each recursive call must be itself the last computation in the function body, rather than placed inside other computations. Here is an example of tail-recursive code:

```scala
def lengthT(s: Seq[Int], res: Int): Int =
  if (s.isEmpty)
    res
  else
    lengthT(s.tail, 1 + res)
```

In this code, one of the branches of the if/else returns a fixed value without doing any recursive calls, while the other branch returns the result of a recursive call to `lengthT(...)`. In the code of `lengthT`, recursive calls never occur within any sub-expressions.



Sergei Winitzki

sergei-winitzki-11a6431

It is not a problem that the **recursive call** to `lengthT` has some sub-expressions such as `1 + res` as its arguments, because all these sub-expressions will be computed before `lengthT` is **recursively called**.

The recursive call to `lengthT` is the last computation performed by this branch of the **if**/**else**. A **tail-recursive** function can have many **if**/**else** or **match**/**case** branches, with or without **recursive calls**; but **all recursive calls must be always the last expressions returned**.

The **Scala** compiler has a feature for checking automatically that a function's code is **tail-recursive** : the **@tailrec annotation**. If a function with a **@tailrec annotation** is not **tail-recursive**, or is not **recursive** at all, the program will not compile.

```
@tailrec def lengthT(s: Seq[Int], res: Int): Int =
if (s.isEmpty) res
else lengthT(s.tail, 1 + res)
```

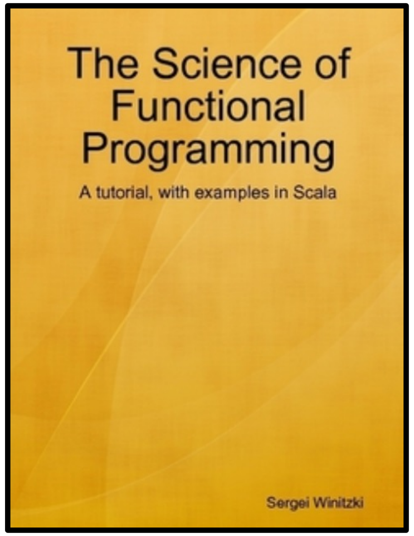
```
def lengthS(s: Seq[Int]): Int =
   if (s.isEmpty) 0
   else 1 + lengthS(s.tail)
```

Let us trace the evaluation of this function on an example:

```
  lengthT(Seq(1,2,3), 0)
= lengthT(Seq(2,3), 1 + 0) // = lengthT(Seq(2,3), 1)
= lengthT(Seq(3), 1 + 1)   // = lengthT(Seq(3), 2)
= lengthT(Seq(), 1 + 2)    // = lengthT(Seq(), 3)
= 3
```

All sub-expressions such as `1 + 1` **and** `1 + 2` **are computed before recursive calls to** `lengthT`. **Because of that, sub-expressions do not grow within the stack memory. This is the main benefit of tail recursion.**

How did we rewrite the code of `lengthS` to obtain the **tail-recursive** code of `lengthT`? An important difference between `lengthS` and `lengthT` is the additional argument, **res**, called the **accumulator argument**. **This argument is equal to an intermediate result of the computation**. The next **intermediate result** (`1 + res`) is computed and passed on to the next **recursive call** via the **accumulator argument**. In the **base case** of the **recursion**, the function now returns the **accumulated result**, **res**, rather than 0, because at that time the computation is finished. **Rewriting code by adding an accumulator argument to achieve tail recursion is called the accumulator technique or the "accumulator trick"**.



Sergei Winitzki

sergei-winitzki-11a6431

It turns out that in **Haskell**, a **left fold** done using **foldl** does not use **constant space**, but rather it uses an amount of **space** that is **proportional** to the length of the list!

See the next slide for how **Richard Bird** describe the problem.

7.5 Controlling Space

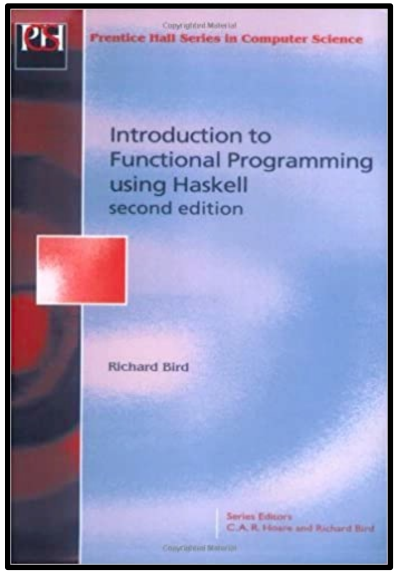
Consider **reduction** of the term $sum\ [1\ ..\ 1000]$, where $sum = foldl\ (+)\ 0$:

$$
\begin{aligned}
& sum\ [1\ ..\ 1000] \\
=\ & foldl\ (+)\ 0\ [1\ ..\ 1000] \\
=\ & foldl\ (+)\ (0{+}1)\ [2\ ..\ 1000] \\
=\ & foldl\ (+)\ ((0{+}1){+}2)\ [3\ ..\ 1000] \\
\vdots \\
=\ & foldl\ (+)\ (\ldots\ (0{+}1){+}2)\ +\ \ldots\ +\ 1000)\ [\,] \\
=\ & (\ldots\ (0{+}1){+}2)\ +\ \ldots\ +\ 1000) \\
=\ & 500500
\end{aligned}
$$

The point to notice is that in computing $sum\ [1\ ..\ n\ ]$ by the **outermost reduction** the expressions grow in size proportional to n. On the other hand, if we use a judicious mixture of **outermost** and **innermost reduction steps**, then we obtain the following **reduction sequence**:

$$
\begin{aligned}
& sum\ [1\ ..\ 1000] \\
=\ & foldl\ (+)\ 0\ [1\ ..\ 1000] \\
=\ & foldl\ (+)\ (0{+}1)\ [2\ ..\ 1000] \\
=\ & foldl\ (+)\ 1\ [2\ ..\ 1000] \\
=\ & foldl\ (+)\ (1{+}2)\ [3\ ..\ 1000] \\
=\ & foldl\ (+)\ (3)\ [3\ ..\ 1000] \\
\vdots \\
=\ & foldl\ (+)\ 500500\ [\,] \\
=\ & 500500
\end{aligned}
$$

The maximum size of any expression in this **sequence** is bounded by a constant. In short, **reducing** to **normal form** by purely **outermost reduction** requires $\Omega(n)$ space, while a combination of **innermost** and **outermost reduction** requires only $O(1)$ space.



Richard Bird

In the case of a function that is **strict** in its first argument however, it is possible to to do a **left fold** that uses **constant space** by using a **strict** variant of **foldl**.

See the next slide for how **Richard Bird** describes the **strict** version of **foldl**.

7.51 Head-normal form and the function strict

Reduction order may be controlled by use of a special function $strict$. A term of the form $strict\ f\ e$ is reduced first by reducing e to **head-normal from**, and then applying f. An expression e is in **head-normal form** if e is a function or if e takes the form of a datatype constructor applied to zero or more arguments. Every expression in **normal form** is in **head-normal form**, but not vice-versa. For example, $e_1 : e_2$ is in **head-normal form** but is in **normal form** only when e_1 and e_2 are both in **normal form**. Similarly $Fork\ e_1\, e_2$ and (e_1, e_2), are in **head-normal form** but are not in **normal form** unless e_1 and e_2 are in **normal form**. In the expression $strict\ f\ e$, the term e will itself be reduced by **outermost reduction**, except, of course, if further calls of $strict$ appear while reducing e.

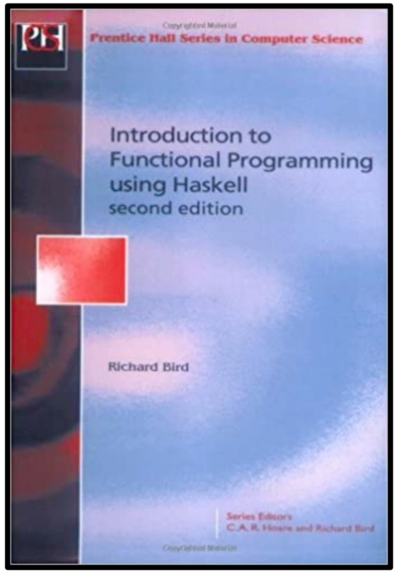
As a simple example, let $succ\ x = x + 1$. Then

$$
\begin{aligned}
& succ\ (succ\ (8 \times 5)) \\
= {} & succ\ (8 \times 5) + 1 \\
= {} & ((8 \times 5) + 1) + 1 \\
= {} & (40 + 1) + 1 \\
= {} & 41 + 1 \\
= {} & 42
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& strict\ succ\ (strict\ succ\ (8 \times 5)) \\
= {} & strict\ succ\ (strict\ succ\ 40) \\
= {} & strict\ succ\ (succ\ 40) \\
= {} & strict\ succ\ (40 + 1) \\
= {} & strict\ succ\ (41) \\
= {} & succ\ (41) \\
= {} & 41 + 1 \\
= {} & 42
\end{aligned}
$$

Both cases perform the same **reduction steps**, but in a different order. Currying applies to $strict$ as to anything else. From this it



Richard Bird

follows that if f is a function of three arguments, writing $strict\ (f\ e_1)\ e_2\ e_3$ causes the second argument e_2 to be reduced early, but not the first or third.

Given this, we can define a function $sfoldl$, a **strict** version of $foldl$, as follows:

$$\begin{aligned} sfoldl\ (\oplus)\ a\ [\,] &= a \\ sfoldl\ (\oplus)\ a\ (x{:}\,xs) &= strict\ (sfoldl\ (\oplus))\ (a \oplus x)\ xs \end{aligned}$$

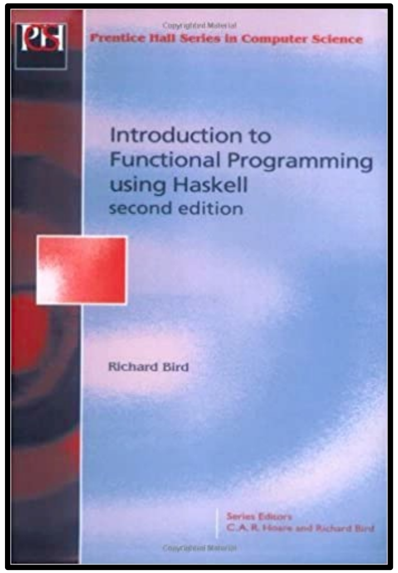
With $sum = sfoldl\ (+)\ 0$ we now have

$$\begin{aligned} & sum\ [1\,..\,1000] \\ =\ & sfoldl\ (+)\ 0\ [1\,..\,1000] \\ =\ & strict\ (sfoldl\ (+))\ (0{+}1)\ [2\,..\,1000] \\ =\ & sfoldl\ (+)\ 1\ [2\,..\,1000] \\ =\ & strict\ (sfoldl\ (+))\ (1{+}2)\ [3\,..\,1000] \\ =\ & sfoldl\ (+)\ 3\ [3\,..\,1000] \\ & \vdots \\ =\ & sfoldl\ (+)\ 500500\ [\,] \\ =\ & 500500 \end{aligned}$$

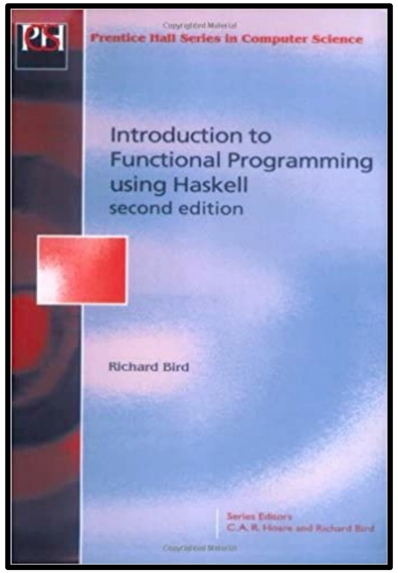
This **reduction sequence** evaluates sum in **constant space**.
…
The operational definition of strict can be re-expressed in the following way:

$$strict\ f\ x = \ \ \mathbf{if}\ \ x = \bot\ \mathbf{then}\ \bot\ \mathbf{else}\ f\ x$$

Recall that a function f is said to be **strict** if $f\ \bot = \bot$. It follows from the above equation that $f = strict\ f$ if and only if f is a **strict** function. To see this, just consider the values of $f\ x$ and $strict\ f\ x$ in the two cases $x = \bot$ and $x \neq \bot$. This explains the name $strict$.



Richard Bird



Furthermore, if f is **strict**, but not everywhere $\bot$, and $e \neq \bot$, then **reduction** of $f\ e$ eventually entails **reduction** of e. Thus, if f is a **strict** function, evaluation of $f\ e$ and $strict\ f\ e$ perform the same **reduction steps**, though possibly in a different **order**. In other words, when f is **strict**, replacing it by $strict\ f$ does not change the meaning or the **asymptotic time** required to apply it, although it may change the **space** required by the computation.

It it easy to show that if $\bot \oplus x = \bot$ for every x, then $foldl\ (\oplus)\ \bot\ xs = \bot$ for every **finite** list xs. In other words, if $(\oplus)$ is **strict** in its left argument, then $foldl\ (\oplus)$ is **strict**, and so is equivalent to $strict\ (foldl\ (\oplus))$, and hence also equivalent to $sfoldl\ (\oplus)$. It follows that replacing $foldl$ by $sfoldl$ in the definition of sum is valid, and the same replacement is valid whenever $foldl$ is applied to a binary operation that is **strict** in its first argument.

Richard Bird

It turns out, as we'll see later, that if $\oplus$ is **strict** in both arguments, and can be computed in $O(1)$ **time** and $O(1)$ **space**, then instead of computing $foldl\ (\oplus)\ e\ xs$, which requires $O(n)$ **time** and $O(n)$ **space** to compute (where n is the length of xs), we can compute $sfoldl\ (\oplus)\ e\ xs$ which, while still requiring $O(n)$ **time**, only requires $O(1)$ **space**.

$$
\begin{aligned}
& sum\ [1..1000] \\
&= foldl\ (+)\ 0\ [1..1000] \\
&= foldl\ (+)\ (0{+}1)\ [2..1000] \\
&= foldl\ (+)\ ((0{+}1){+}2)\ [3..1000] \\
&\ \vdots \\
&= foldl\ (+)\ (\ldots (0{+}1){+}2) + \ldots + 1000)\ [\,] \\
&= (\ldots (0{+}1){+}2) + \ldots + 1000) \\
&= 500500
\end{aligned}
$$

We saw earlier that the reason why $foldl$ requires $O(n)$ **space** is that it builds a growing intermediate expression that only gets reduced once the whole list has been traversed. For this reason, $foldl$ can't possibly be using an **accumulator** for the customary purpose of maintaining a running **intermediate result** so that only constant **space** is required.

@philip_schwarz

Also, where is that **intermediate expression** stored? While it makes sense to store it in **heap** memory, why is it that earlier, when we computed $foldl$ (+) `0` [1 100,000,000], it resulted in a **stack overflow exception**? It looks like $foldl$ can't possibly be using **tail–recursion** for the customary purpose of avoiding **stack overflows**. The next two slides begin to answer these questions.

Left Folds, Laziness, and Space Leaks

To keep our initial discussion simple, we use `foldl` throughout most of this section. This is convenient for testing, but we will never use `foldl` in practice. The reason has to do with Haskell's nonstrict evaluation. If we apply `foldl (+) [1,2,3]`, it evaluates to the expression `(((0 + 1) + 2) + 3)`. We can see this occur if we revisit the way in which the function gets expanded:

```
   foldl (+) 0                   (1:2:3:[])
== foldl (+) (0 + 1)             (2:3:[])
== foldl (+) ((0 + 1) + 2)       (3:[])
== foldl (+) (((0 + 1) + 2) + 3) []
==           (((0 + 1) + 2) + 3)
```

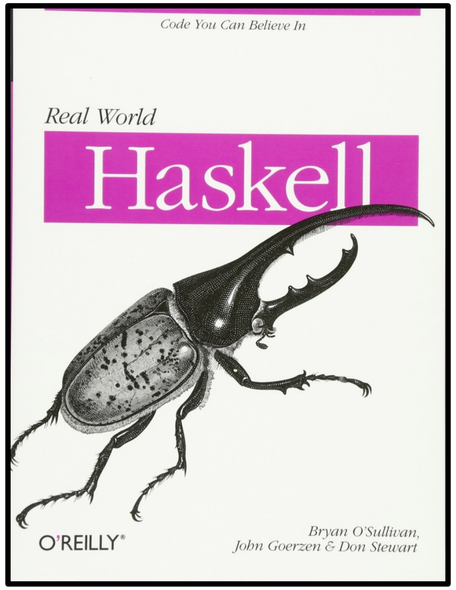
The final expression will not be evaluated to 6 until its value is demanded. Before it is evaluated, it must be stored as a thunk. Not surprisingly, a thunk is more expensive to store than a single number, and the more complex the thunked expression, the more space it needs. For something cheap such as arithmetic, thunking an expression is more computationally expensive than evaluating it immediately. We thus end up paying both in space and in time.
When GHC is evaluating a thunked expression, it uses an internal stack to do so. Because a thunked expression could potentially be infinitely large, GHC places a fixed limit on the maximum size of this stack. Thanks to this limit, we can try a large thunked expression in *ghci* without needing to worry that it might consume all the memory:

```
ghci> foldl (+) 0 [1..1000]
500500
```

From looking at this expansion, we can surmise that this creates a thunk that consists of 1,000 integers and 999 applications of (+). That's a lot of memory and effort to represent a single number! With a larger expression, although the size is still modest, the results are more dramatic:

```
ghci> foldl (+) 0 [1..1000000]
*** Exception: stack overflow
```

On small expressions, `foldl` will work correctly but slowly, due to the thunking overhead that it incurs.



Bryan O'Sullivan
John Goerzen
Donald Bruce Stewart

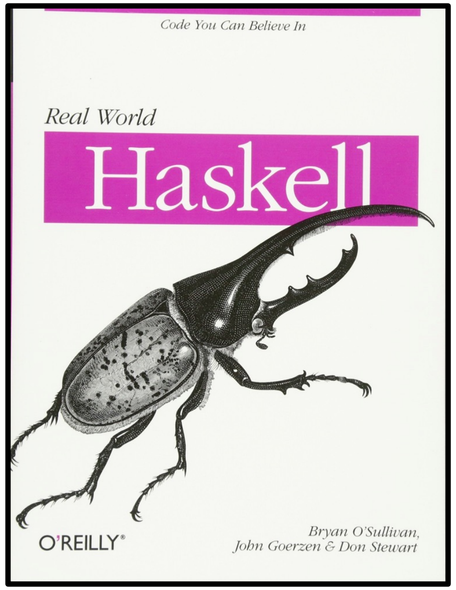
We refer to this invisible thunking as a *space leak*, because our code is operating normally, but it is using far more memory than it should.

On larger expressions, code with a space leak will simply fail, as above. A space leak with foldl is a classic roadblock for new Haskell programmers. Fortunately, this is easy to avoid.

The `Data.List` module defines a function named foldl' that is similar to foldl, but does not build up thunks. The difference in behavior between the two is immediately obvious:

```
ghci> foldl (+) 0 [1..1000000]
*** Exception: stack overflow
ghci> :module +Data.List
ghci> foldl' (+) 0 [1..1000000]
500000500000
```

Due to `foldl`'s thunking behavior, it is wise to avoid this function in real programs, even if it doesn't fail outright, it will be unnecessarily inefficient. Instead, import `Data.List` and use `foldl'`.



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So the **sfoldl** function described by **Richard Bird** is called **foldl'**.

That explanation clears up things a lot. The next slide reinforces it and complements it nicely.

Performance/Strictness

https://wiki.haskell.org/Performance/Strictness

Haskell is a **non-strict** language, and most implementations use a strategy called ***laziness*** to run your program. Basically **laziness** == **non-strictness** + **sharing**.

Laziness can be a useful tool for improving performance, but more often than not it reduces performance by adding a constant overhead to everything. **Because of laziness, the compiler can't evaluate a function argument and pass the value to the function, it has to record the expression in the heap in a *suspension* (or thunk) in case it is evaluated later**. Storing and evaluating suspensions is costly, and unnecessary if the expression was going to be evaluated anyway.

Strictness analysis

Optimising compilers like GHC try to reduce the cost of laziness using ***strictness analysis***, which attempts to determine which function arguments are always evaluated by the function, and hence can be evaluated by the caller instead…

The common case of misunderstanding of **strictness analysis** is when folding (reducing) lists. If this program

```
main = print (foldl (+) 0 [1..1000000])
```

is compiled in GHC without "-O" flag, **it uses a lot of heap and stack…** Look at the definition from the standard library:

```
foldl :: (a -> b -> a) -> a -> [b] -> a
foldl f z0 xs0 = lgo z0 xs0
  where lgo z [] = z
        lgo z (x:xs) = lgo (f z x) xs
```

`lgo`, **instead of adding elements of the long list, creates a thunk for** `(f z x)`. `z` **is stored within that thunk, and** `z` **is a thunk also, created during the previous call to** `lgo`. **The program creates the long chain of thunks. Stack is bloated when evaluating that chain.**
With "-O" flag GHC performs strictness analysis, then it knows that `lgo` **is strict in** `z` **argument, therefore thunks are not needed and are not created**.

So the reason why we see all that **memory consumption** in the first two scenarios below is that **foldl** creates a huge chain of **thunks** that is the **intermediate expression** representing the sum of all the list's elements.

The reason why the **stack overflows** in the second scenario is that it is not large enough to permit evaluation of the **final expression**.

The reason why the **fold** does not terminate in the third scenario is that since the list is **infinite**, **foldl** never finishes building the **intermediate expression**. The reason it does not **overflow** the **stack** is that it doesn't even use the **stack** to evaluate the **final expression** since it never finishes building the expression.

Left Fold Scenario	Code	Result	Approximate Duration
List Size: large Function Strictness: strict in **both** arguments	`foldl (+) 0 [1..10,000,000]`	50000005000000	4 seconds Ghc memory: • initial: 27.3MB • final: 1.10 GB
List Size: huge Function Strictness: strict in **both** arguments	`foldl (+) 0 [1..100,000,000]`	**** Exception: stack overflow*	3 seconds Ghc memory: • initial: 27.3MB • final: 10 GB
List Size: infinite Function Strictness: strict in **both** arguments	`foldl (+) 0 [1..]`	Does not terminate Keeps going	I stopped it after 3 min Ghc memory: • initial: 27.3MB • final: 22 GB

In the next slide we see **Haskell** expert **Michael Snoyman** make the point that **foldl** is broken and that **foldl'** is the **one true left fold**.

@philip_schwarz

foldl

Duncan Coutts already did this one. **foldl is broken**. It's a **bad function**. **Left folds are supposed to be strict, not lazy**. End of story. Goodbye. **Too many space leaks have been caused by this function**. **We should gut it out entirely**.

But wait! A **lazy left fold** makes perfect sense for a Vector! Yeah, no one ever meant that. And **the problem isn't the fact that this function exists. It's the name**. **It has taken the hallowed spot of the One True Left Fold**. **I'm sorry, the One True Left Fold is strict**.

Also, side note: we can't raise linked lists to a position of supreme power within our ecosystem and then pretend like we actually care about vectors. We don't, we just pay lip service to them. Until we fix the wart which is overuse of lists, **foldl** is only ever used on lists.
OK, back to this **bad left fold**. This is all made worse by the fact that **the true left fold, foldl', is not even exported by the Prelude**. **We Haskellers are a lazy bunch. And if you make me type in import Data.List** (**foldl'**), **I just won't. I'd rather have a space leak than waste precious time typing in those characters.**

Alright, so what should you do? **Use an alternative prelude that doesn't export a bad function**, **and does export a good function**. **If you really, really want a lazy left fold: add a comment, or use a function named foldlButLazyIReallyMeanIt. Otherwise I'm going to fix your code during my code review**.

Michael Snoyman
@snoyberg

True mastery of Haskell comes down to knowing which things in core libraries should be avoided like the plague.

* foldl
* sum/product
* Data.Text.IO
* Control.Exception.bracket (use unliftio instead, handles interruptible correctly)

Just as some examples

11:20 AM · Oct 27, 2020

76 18 people are Tweeting about this

Michael Snoyman
@snoyberg

Following up on the discussion here yesterday: Haskell: The Bad Parts, part 1. Featuring bracket, sum, product, foldl, and Data.Text.IO. Ideas for future posts are welcome!

Haskell: The Bad Parts, part 1
The first part of a blog post series on the parts of Haskell we should avoid using.
snoyman.com

9:38 AM · Oct 28, 2020 · Twitter for iPhone

15 Retweets **3** Quote Tweets **85** Likes

We said earlier that if $\oplus$ is **strict** in both arguments, and can be computed in $O(1)$ **time** and $O(1)$ **space**, then instead of computing $foldl\ (\oplus)\ e\ xs$, which requires $O(n)$ **time** and $O(n)$ **space** to compute (where n is the length of xs), we can compute $sfoldl\ (\oplus)\ e\ xs$ which, while still requiring $O(n)$ **time**, only requires $O(1)$ **space**.

This is explained by **Richard Bird** in the next slide, in which he makes some other very useful observations as he revisits **fold**.

Remember earlier, when we looked in more detail at **Tony Morris**' example in which he **folds** an **infinite list** of booleans using (&&)? That is also covered in the next slide.

```
λ> :{
(&&) False x = False
(&&) True x = x
:}
λ> infinity = False : infinity
λ> conjunct = foldr (&&) True
```

```
λ> infinity
[False,False,False,False,False,False,False,False
,False,False,False,False,False,False, etc, etc
```

$conjunct\ infinity$

= { definition of $conjunct$ }

$foldr\ \ (\&\&)\ \ True\ infinity$

= { definition of $infinity$ }

$foldr\ \ (\&\&)\ \ True\ \ (False : infinity)$

= { definition of $foldr$ }

$False\ \&\&\ \ (foldr\ \ (\&\&)\ \ True\ infinity)$

= { definition of && }

$False$

```
λ> conjunct infinity
False
```

Lazy evaluation causes just enough of *infinity* to be valuated to allow *foldr* to be invoked.

$$\begin{array}{ll} foldr & :: (\alpha \to \beta \to \beta) \to \beta \to [\alpha] \to \beta \\ foldr\ f\ e\ [\,] & = e \\ foldr\ f\ e\ (x{:}xs) & = f\ x\ (foldr\ f\ e\ xs) \end{array}$$

$$\begin{array}{ll} (\&\&) & :: Bool \to Bool \to Bool \\ False\ \&\&\ x & = False \\ True\ \ \&\&\ x & = x \end{array}$$

```
λ> False && undefined
False
```

$$\begin{array}{ll} infinity & :: [Bool] \\ infinity & = False : infinity \end{array}$$

$$\begin{array}{ll} conjunct & :: [Bool] \to Bool \\ conjunct & = foldr\ (\&\&)\ True \end{array}$$

```
λ> conjunct (False:undefined)
False
```

(&&) is **strict** in its second argument only when its first argument is True.
conjunct is **strict** in its second argument only when it is a list of True values.

```
λ> infinity = True : infinity
λ> conjunct infinity
{ Interrupted }
```

7.5.2 Fold revisited

The **first duality theorem** states that if $(\oplus)$ is **associative** with **identity** e, then

$$foldr\ (\oplus)\ e\ xs = foldl\ (\oplus)\ e\ xs$$

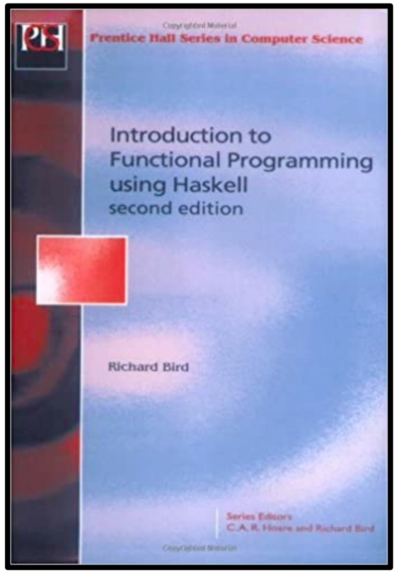
for all **finite** lists xs. On the other hand, the two expressions may have different **time** and **space complexities**. Which one to use depends on the properties of $(\oplus)$.

First, suppose that $\oplus$ is **strict** in both arguments, and can be computed in $O(1)$ time and $O(1)$ space. Examples that fall into this category are $(+)$ and $(\times)$. In this case it is not hard to verify that $foldr\ (\oplus)\ e$ and $foldl\ (\oplus)\ e$ both require $O(n)$ **time** and $O(n)$ **space** to compute a list of length n. However, the same argument used above for sum generalizes to show that, in this case, $foldl$ may safely be replaced by $sfoldl$. While $sfoldl\ (\oplus)\ e$ still requires $O(n)$ **time** to evaluate on a list of length n, it only requires $O(1)$ **space**. So in this case, $sfoldl$ is the clear winner.

If $\oplus$ does not satisfy the above properties, then choosing a winner may not be so easy. A good rule of thumb, though, is that if $\oplus$ is **nonstrict** in either argument, then $foldr$ is usually more efficient than $foldl$. We saw one example in section 7.2: the function $concat$ is more efficiently computed using $foldr$ than using $foldl$. Observe that while $+\!\!+$ is **strict** in its first argument, it is not **strict** in its second.

Another example is provided by the function $and = foldr\ (\wedge)\ True$. Like $+\!\!+$, the operator $\wedge$ is **strict** in its first argument, but **nonstrict** in its second. In particular, $False\ \wedge\ x$ returns without evaluating x. Assume we are given a list xs of n boolean values and k is the first value for which $xs\ !!\ k = False$. Then evaluation of $foldr\ (\wedge)\ True\ xs$ takes $O(k)$ **steps**, whereas $foldl\ (\wedge)\ True\ xs$ requires $\Omega(n)$ steps. Again, $foldr$ is a better choice.

To summarise: for functions such as $+$ or $\times$, that are **strict** in both arguments and can be computed in constant **time** and **space**, $sfoldl$ is more efficient. But for functions such as $\wedge$ and and $+\!\!+$, that are **nonstrict** in some argument, $foldr$ is often more efficient.



Richard Bird

@philip_schwarz

Here is how **Richard Bird** defined a **strict left fold**:

$$
\begin{aligned}
&sfoldl\ (\oplus)\ a\ [\,] &&= a \\
&sfoldl\ (\oplus)\ a\ (x\!:\!xs) &&= strict\ (sfoldl\ (\oplus))\ (a \oplus x)\ xs
\end{aligned}
$$

As we saw earlier, these days the **strict left fold** function is called **foldl’**. How is it defined?

To answer that, we conclude this slide deck by going through sections of **Graham Hutton**’s explanation of **strict application**.

15.7 Strict Application

Haskell uses **lazy evaluation** by default, but also provides a special ***strict*** version of function application, written as `$!`, which can sometimes be useful. Informally, **an expression of the form** `f $! x` **behaves in the same way as the normal functional application** `f x`, **except that the top-level of evaluation of the argument expression** `x` **is forced before the function** `f` **is applied**.

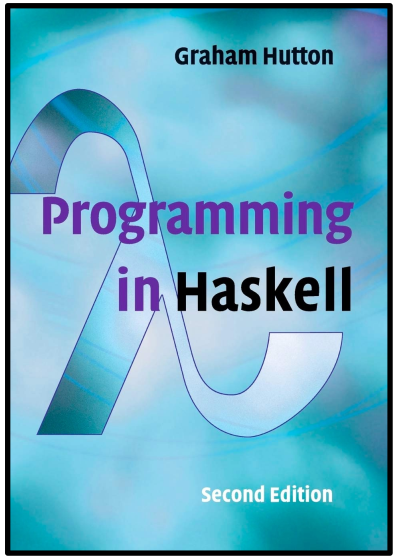
…

In **Haskell**, **strict** application is mainly used to improve the **space performance** of programs. For example, consider a function **sumwith** that calculates the sum of a list of integers using an **accumulator value**:

```
sumwith :: Int -> [Int] -> Int
sumwith v [] = v
sumwith v (x:xs) = sumwith (v+x) xs
```

Then, using **lazy evaluation**, we have:

```
  sumwith 0 [1,2,3]
=    { applying sumwith }
  sumwith (0+1) [2,3]
=    { applying sumwith }
  sumwith ((0+1)+2) [3]
=    { applying sumwith }
  sumwith (((0+1)+2)+3) []
=    { applying sumwith }
  ((0+1)+2)+3
=    { applying the first + }
  (1+2)+3
=    { applying the first + }
  3+3
=    { applying + }
  6
```



Graham Hutton
@haskellhutt

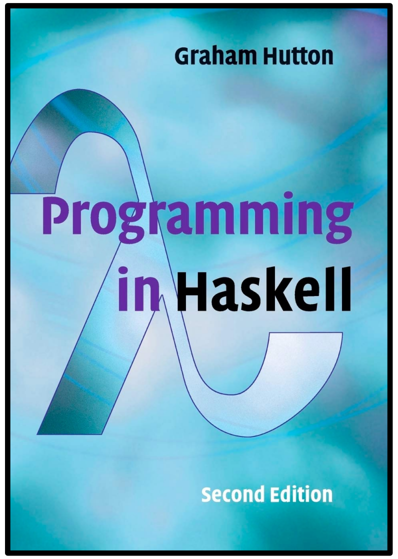
Note that the entire summation `((0+1)+2)+3` is constructed before any of the component additions are actually performed. More generally, `sumwith` will construct a summation whose size is proportional to the number of integers in the original list, which for a long list may require a significant amount of space. In practice, it would be preferable to perform each addition as soon as it is introduced, to improve the space performance of the function.

This behaviour can be achieved by redefining `sumwith` using **strict application**, to force evaluation of its **accumulator value**:

```
sumwith v [] = v
sumwith v (x:xs) = (sumwith $! (v+x)) xs
```

For example, we now have:

```
  sumwith 0 [1,2,3]
=    { applying sumwith }
  (sumwith $! (0+1)) [2,3]
=    { applying + }
  (sumwith $! 1) [2,3]
=    { applying $! }
  sumwith 1 [2,3]
=    { applying sumwith }
  (sumwith $! (1+2)) [3]
=    { applying + }
  (sumwith $! 3) [3]
=    { applying $! }
  sumwith 3 [3]
=    { applying sumwith }
  (sumwith $! (3+3)) []
=    { applying + }
  (sumwith $! 6) []
```



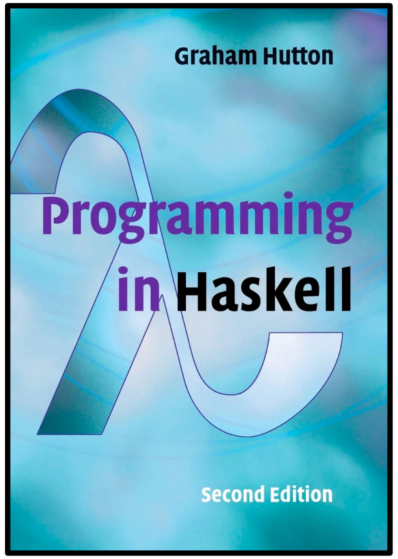
Graham Hutton
@haskellhutt

```
=    { applying $! }
  sumwith 6 []
=    { applying sumwith }
  6
```

This evaluation requires more steps than previously, due to the additional overhead of using strict application, but now performs each addition as soon as it is introduced, rather than constructing a large summation. Generalising from the above example, the library `Data.Foldable` provides a strict version of the higher-order library function foldl that forces evaluation of its accumulator prior to processing the tail of the list:

```
foldl' :: (a -> b -> a) -> a -> [b] -> a
foldl' f v [] = v
foldl' f v (x:xs) = ((foldl' f) $! (f v x)) xs
```

For example, using this function we can define `sumwith` = **foldl'** (+). It is important to note, however, that **strict application** is not a silver bullet that automatically improves the space behaviour of **Haskell** programs. Even for relatively simple examples, the use of **strict application** is a specialist topic that requires careful consideration of the behaviour of **lazy evaluation**.



Graham Hutton
@haskellhutt

That’s all for Part 5. I hope you found that useful.

See you in Part 6.