# Function Applicative for Great Good of Palindrome Checker Function

## Polyglot FP for Fun and Profit – Haskell and Scala

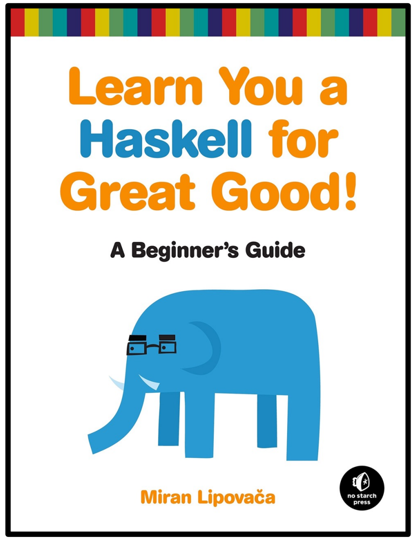
Embark on an informative and fun journey through everything you need to know

to understand how the **Applicative instance** for **functions**

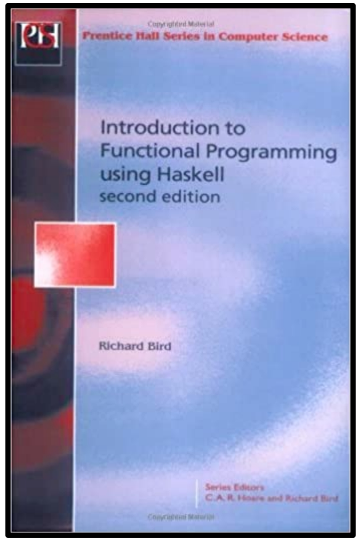
makes for a terse **palindrome checker** function definition in **point-free style**

Impure Pics
@impurepics

Ἑκάτη
@TechnoEmpress




Miran Lipovača




Richard Bird

Amar Shah
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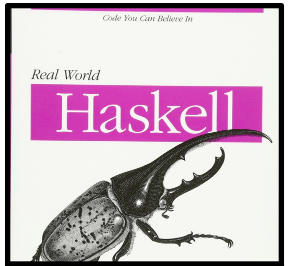


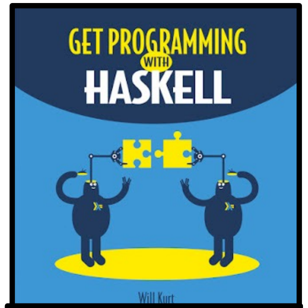


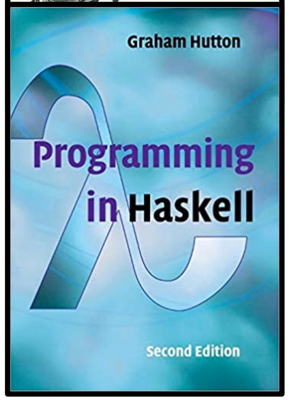


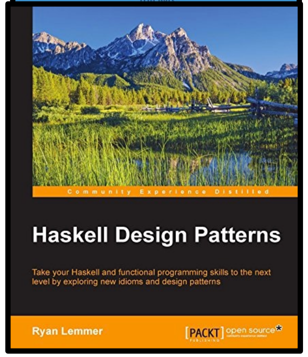


slides by

@philip_schwarz slideshare https://www.slideshare.net/pjschwarz

The initial motivation for this slide deck is an **Applicative palindrome checker** illustrated collaboratively by **Impure Pics** and **Ἑκάτη**.

@philip_schwarz

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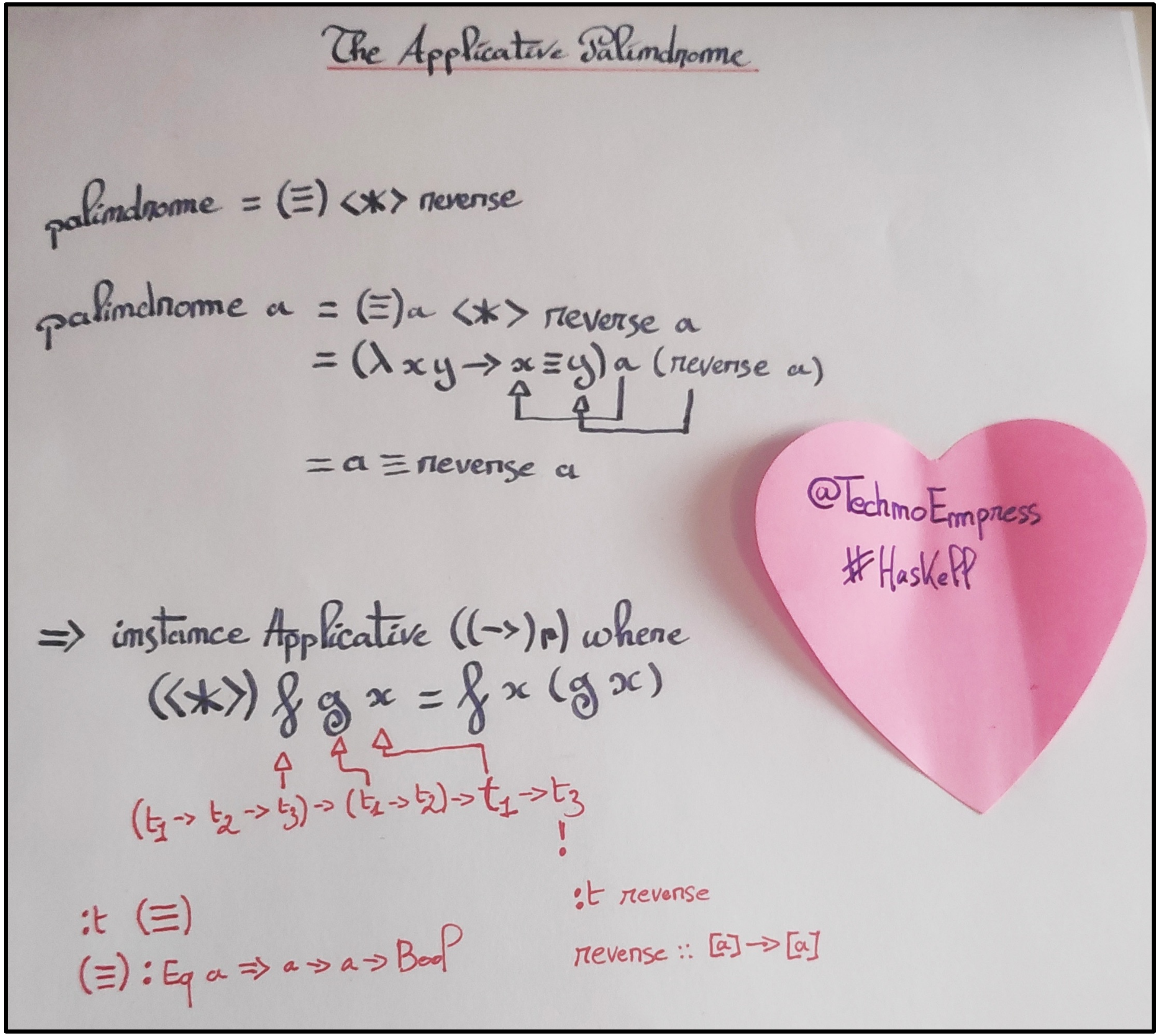
**Ἑκάτη** @TechnoEmpress · Dec 9

I am glad to present to you all an exclusive collaboration with @impurepics : This beautiful illustration of the "Applicative palindrome checker" #Haskell party trick.

THE APPLICATIVE PALINDROME

```
palindrome = (≡) <*> reverse

palindrome a = (≡) a <*> reverse a
             = (λxy → x ≡ y) a (reverse a)
             = a ≡ reverse a

⇒ instance Applicative ((→) r) where
     (<*>) f g x = f x (g x)

  (t1 → t2 → t3) → (t1 → t2) → t1 → t3

:t (≡)                          :t reverse
(≡) :: Eq a ⇒ a → a → Bool      reverse :: [a] → [a]
```

Here is a handwritten version of the illustration, by Ἑκάτη.


https://ko-fi.com/hecatemoonlight/gallery#galleryItemView

One of the things that I did when I first saw the **palindrome** function was ask myself: what is (≡) ?

```
palindrome = (≡) <*> reverse
```

In the illustration by **Impure Pics** and **Ἑκάτη** we see that (≡) has the following type

```
:t (≡)
(≡) :: Eq a => a -> a -> Bool
```

But that type is the same as the type of the (==) function

```
λ :type (==)
(==) :: Eq a => a -> a -> Bool
```

So it looks like (≡) is just an alias for (==)

```
λ (≡) = (==)
```

https://en.wikipedia.org/wiki/Triple_bar

The **triple bar**, ≡, is a symbol with multiple, context-dependent meanings.

It has the appearance of an equals sign ⟨=⟩ sign with a third line.

The triple bar character in Unicode is code point `U+2261` ≡ IDENTICAL TO

…

In mathematics, the triple bar is sometimes used as a symbol of identity or an equivalence relation.

Another thing that I did when I first saw the **palindrome** function is notice that it is written in **point-free style**.

```
palindrome = (≡) <*> reverse
```

See the next nine slides for a quick refresher on (or introduction to) **point-free style**. Feel free to skip the slides if you are already up to speed on the subject.

**1.4.1 Extensionality**

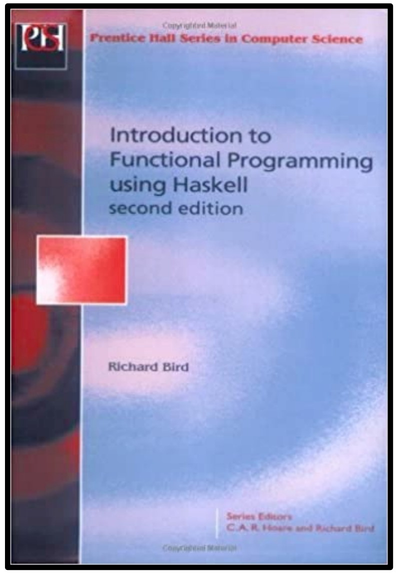
Two functions are equal if they give equal results for equal arguments. Thus, $f = g$ if and only if $f\ x = g\ x$ for all $x$. This principle is called the principle of ***extensionality***. It says that the important thing about a function is the correspondence between arguments and results, not how this correspondence is described.

For instance, we can define the function which doubles its argument in the following two ways:

$$\begin{array}{lcl} double, double' & :: & Integer \rightarrow Integer \\ double\ x & = & x + x \\ double'\ x & = & 2 \times x \end{array}$$

The two definitions describe different ***procedures*** for obtaining the correspondence, one involving addition and the other involving multiplication, but $double$ and $double'$ define the same function value and we can assert $double = double'$ as a mathematical truth. Regarded as procedures for evaluation, one definition may be more or less 'efficient' than the other, but the notion of efficiency is not one that can be attached to function values themselves. This is not to say, of course, that efficiency is not important; after all, we want expressions to be evaluated in a reasonable amount of time. The point is that efficiency is an ***intensional*** property of definitions, not an ***extensional*** one.

***Extensionality*** means that we can prove $f = g$ by proving that $f\ x = g\ x$ for all $x$. Depending on the definitions of $f$ and $g$, we may also be able to prove $f = g$ directly. **The former kind of proof is called an *applicative* or *point-wise* style of proof, while the latter is called a *point-free* style**.




Richard Bird

### 1.4.7 Functional composition

The **composition** of two functions $f$ and $g$ is denoted by $f\ .\ g$ and is defined by the equation

$$(\,.\,) \quad :: (\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$$
$$(f\ .\ g)\ x \ = f\ (g\ x)$$

... In words, $f\ .\ g$ applied to $x$ is defined to be the outcome of first applying $g$ to $x$, and then applying $f$ to the result. Not every pair of functions can be composed since the types have to match up: we require that $g$ has type $g :: \alpha \to \beta$ for some types $\alpha$ and $\beta$, and that $f$ has type $f :: \beta \to \gamma$ for some type $\gamma$. Then we obtain $f.g :: \alpha \to \gamma$. For example, given $square :: Integer \to Integer$, we can define

$$quad :: \ Integer \to Integer$$
$$quad = square\ .\ square$$

By the definition of **composition**, this gives exactly the same function $quad$ as

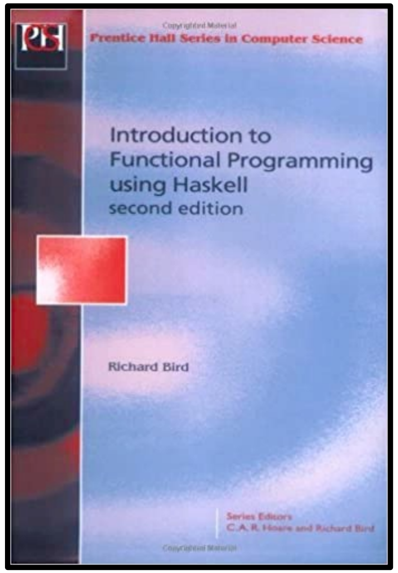
$$quad \ :: \ Integer \to Integer$$
$$quad\ x = square\ (square\ x)$$

This example illustrates **the main advantage of using function composition in programs: definitions can be written more concisely. Whether to use a point-free style or a point-wise style is partly a question of taste**... However, whatever the style of expression, it is good practice to construct complicated functions as the **composition** of simpler ones.

Functional composition is an associative operation. We have

$$(f\ .\ g)\ .\ h \ = f\ .\ (g\ .\ h)$$

For all functions $f$, $g$, and $h$ of the appropriate types. Accordingly, there is no need to put in parentheses when writing sequences of compositions.




Richard Bird

**Composing functions**

Let's **compose** these three functions, f, g, and h, in a few different ways:

```
f, g, h :: String -> String
```

The most rudimentary way of **combining** them is through **nesting**:

```
z x = f (g (h x))
```

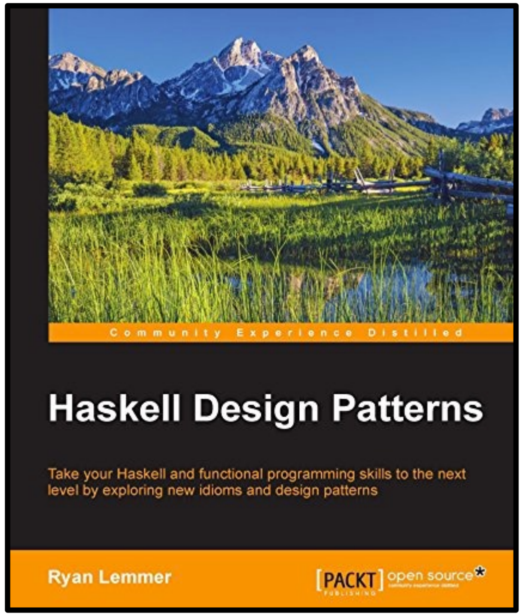
**Function composition** gives us a **more idiomatic way** of **combining** functions:

```
z' x = (f . g . h) x
```

**Finally, we can abandon any reference to arguments**:

```
z'' = f . g . h
```

**This leaves us with an expression consisting of only functions**. **This is the "point-free" form**. **Programming with functions in this style, free of arguments, is called tacit programming**. **It is hard to argue against the elegance of this style, but in practice, point-free style can be more fun to write than to read**.




Ryan Lemmer

## Point-Free Style

Another common use of **function composition** is defining functions in the ***point-free style***. For example, consider a function we wrote earlier:

```
sum' :: (Num a) => [a] -> a
sum' xs = foldl (+) 0 xs
```

The xs is on the far right on both sides of the equal sign. Because of currying, we can omit the xs on both sides, since calling foldl (+) 0 creates a function that takes a list. In this way, we are writing the function in **point-free style**:

```
sum' :: (Num a) => [a] -> a
sum' = foldl (+) 0
```

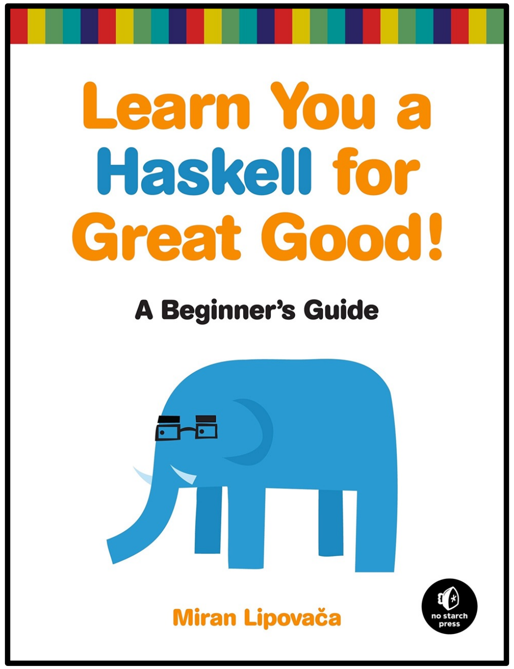
As another example, let's try writing the following function in **point-free style**:

```
fn x = ceiling (negate (tan (cos (max 50 x))))
```

We can't just get rid of the x on both right sides, since the x in the function body is surrounded by parentheses. cos (max 50) wouldn't make sense—you can't get the cosine of a function. What we *can* do is express fn as a **composition** of functions, like this:

```
fn = ceiling . negate . tan . cos . max 50
```

Excellent! **Many times, a point-free style is more readable and concise, because it makes you think about functions and what kinds of functions composing them results in, instead of thinking about data and how it's shuffled around**. You can take simple functions and use composition as glue to form more complex functions.
**However, if a function is too complex, writing it in point-free style can actually be less readable**. **For this reason, making long chains of function composition is discouraged**. The preferred style is to use let bindings to give labels to intermediary results or to split the problem into subproblems that are easier for someone reading the code to understand.




Miran Lipovača

## Pretty Printing a String

When we must pretty print a string value, JSON has moderately involved escaping rules that we must follow. At the highest level, a string is just a series of characters wrapped in quotes:

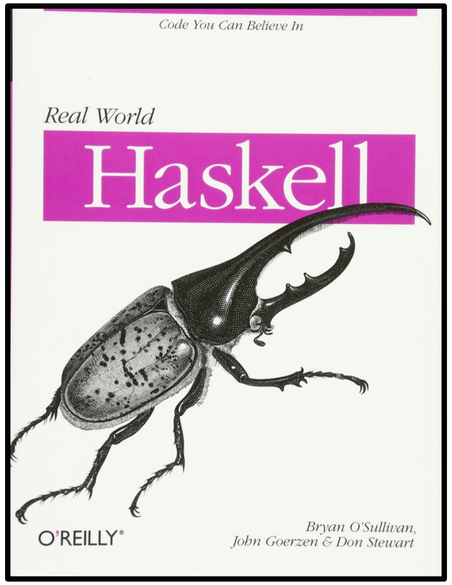
```
string :: String -> Doc
string = enclose '"' '"' . hcat . map oneChar
```

### POINT-FREE STYLE

**This style of writing a definition exclusively as a composition of other functions is called *point-free style*. The use of the word *point* is not related to the "." character used for function composition. The term *point* is roughly synonymous (in Haskell) with *value*, so a *point-free* expression makes no mention of the values that it operates on.**

**Contrast this *point-free* definition of string with this "*pointy*" version, which uses a variable, s, to refer to the value on which it operates:**

```
pointyString :: String -> Doc
pointyString s = enclose '"' '"' (hcat (map oneChar s))
```



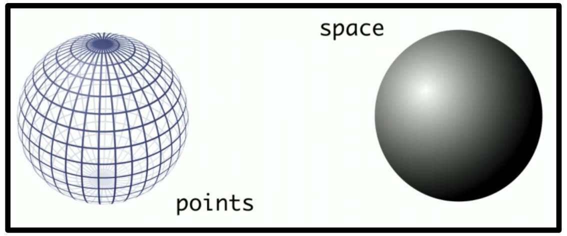
# point-free or die

tacit programming in haskell and beyond

Amar Shah @amar47shah

YouTube GB https://www.youtube.com/watch?v=seVSlKazsNk

So what is this thing, **point-free**? Point-free is **a style of writing function definitions**. A point-free expression, is a kind of function definition. But point free is kind of bigger than that. It's **a way of talking about transformations that emphasizes the space instead of the individual points that make up the space**.




So what's this other thing: **tacit**? So tacit is just **a synomym for quiet**. And **tacit code is quieter than noisy code**, and here is an example of a **point-free definition**, down here at the bottom, and its **pointful** counterpart:

```
sum xs = foldr (+) 0 xs

        ⥮

sum    = foldr (+) 0
```

**pointful** definition

**point-free** definition

So both of these definitions describe the same function. It's the sum function and you could say the first one reads like this: **to take the sum of a list of things, do a right fold over that list, starting at zero and using addition to combine items**. Or you could just say **sum is a right fold, using addition, starting at zero**.

```
but why?
```

But **why would you want to use the point-free definition**? **What's the point of removing all these arguments from your functions**?

```
Point-free style is a tool for tacitness.

Tacitness is a tool for communication.
```

So that means that **when you want to make your code communicate better, you can use tacitness, and you can use point-free style to make it quiet**.

And why would you want to do that? To `calibrate abstraction`

One reason is that **tacit code can keep you talking at the right level of abstraction**, so that you are not constantly switching between low level and high level constructions.

```haskell
lengths ls = map length ls

        ⥮

lengths    = map length
```

Here is another example.

**lengths** can **take a list of lists, and map the length function over that list of lists**, **or you could just say lengths is a map of length**.

Amar Shah
@amar47shah

```
be more expressive
```

So **point-free** definitions are a way that you can be **more expressive** with your code

```haskell
totalNumber ls = sum (lengths ls)

        ⥮

totalNumber    = sum . lengths
```

So here is one more example. Here is a function **totalNumber**, of a list of lists. To get he total number of the items in my sublists, **I want to take the lengths of all my sublists and then I want to add all those together**.

**Or I could just say the total number is the composition of sum and length**.
So I can use the composition operator, right here.

```
totalNumber ls = sum (lengths ls)

⥮

totalNumber    = sum . lengths
```

And then you might say something like:

“but point-free has more *points*!”

so, that little dot, it is not a point, it is **composition** and you are better off thinking of it as a **pipe**.

Amar Shah
@amar47shah

```
outside . inside is a composition

1    \x -> outside (inside x)
2    \x -> outside $ inside x
3    \x -> outside . inside $ x
4          outside . inside

“`outside` composed with `inside`”
```

If I have two functions, **outside** and **inside**, their **composition works like this**:

**Composition is a function that takes an argument, applies inside and then applies outside**.
I can do some Haskell magic here, I can use the $ sign operator here, instead of parentheses, it means the same thing essentially, and I can move the dollar sign to the end here, and replace it with composition. Of course, when I have a lambda abstraction that takes an argument and does nothing but apply that function to the argument, then I don’t even need the lambda abstraction.

So **you can read this as outside composed with inside**. That’s probably the best way to read it if you want to remember that it is a composition.

# when?

So **when should you use a point-free definition**, because you can use definitions that are point-free and you can use ones that aren’t?

```
Use point-free style when it communicates better.


    Avoid point-free style when it doesn’t.
```

And here are my two rules, which are really just one rule:

**Use it when it is good and don’t use it when it is bad**.

Generally, trying to do **point-free style** in **Scala** is a dead end.

**Martin Odersky**
@odersky

**Daniel Spiewak**
@djspiewak

To elaborate on this just a little bit, **inability to effectively encode point-free (in general!) is a fundamental limitation of object-oriented languages**. **Effective point-free style absolutely requires that function parameters are ordered from least-to-most specific**. For example:

```
Prelude> :type foldl
foldl :: (a -> b -> a) -> a -> [b] -> a
```

This ordering is required for **point-free** because it makes it possible to apply a function, specifying a subset of the parameters, receiving a function that is progressively more specific (and less general). If we invert the arguments, then the most specific parameter must come *first*, meaning that the function has lost all generality before any of the other parameters have been specified!

**Unfortunately, most-specific-first is precisely the ordering imposed by any object-oriented language**. Think about this: what is the most specific parameter to the **fold** function? Answer: the list! **In object-oriented languages, dispatch is always defined on the most-specific parameter in an expression**. This is another way of saying that we define the **foldLeft** function on List, *not* on Function2.

Now, it is possible to extend an object-oriented language with reified messages to achieve an effective **point-free** dispatch mechanism (basically, by making it possible to construct a method dispatch prior to having the dispatch receiver), but **Scala** does not do this. In any case, such a mechanism imposes a lot of other requirements, such as a rich structural type system (much richer than **Scala** anonymous interfaces).

**There are certainly special cases where point-free does work in Scala, but by and large, you're not going to be able to use it effectively. Everything in the language is conspiring against you, from the syntax to the type system to the fundamental object-oriented nature itself!**

```
Prelude> :type foldl
foldl :: Foldable t => (b -> a -> b) -> b -> t a -> b
```

```
sealed abstract class List[+A] …
  …
  def foldLeft[B](z: B)(op: (B, A) => B): B
```

After that refresher on (or introduction to) **point-free style**, let's go back to the **palindrome** function and take it for a spin.

By the way, if you liked the '**point free or die**' slides, you can find the rest of that deck here:

https://www2.slideshare.net/pjschwarz/point-free-or-die-tacit-programming-in-haskell-and-beyond

Let's define the equality function (≡). It simply applies equality operator == to its arguments.

```
(≡) :: Eq a => a -> a -> Bool
(≡) x y = x == y
```

Let's try it out

```
λ 3 ≡ 4

False

λ 3 ≡ 3

True
```

```
λ [1,2,3] ≡ [1,3,2]

False

λ [1,2,3] ≡ [1,2,3]

True
```

We can now define the **palindrome** function. A **palindrome** is something that equals its reverse.

```
palindrome :: Eq a => [a] -> Bool
palindrome = (≡) <*> reverse
```

Let's try it out

```
λ palindrome "abcabc"

False

λ palindrome "abccba"

True
```

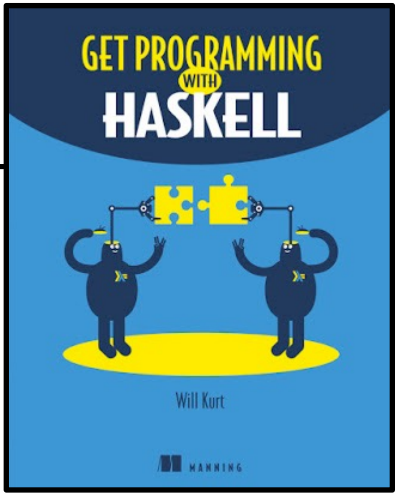
```
λ palindrome [1,2,3,1,2,3]

False

λ palindrome [1,2,3,3,2,1]

True
```

Here is a more interesting palindrome.

```
λ palindrome "A man, a plan, a canal: Panama!"

False

λ clean str = fmap toLower (filter (`notElem` [' ',',',':','!']) str)

λ clean  "A man, a plan, a canal: Panama!"

"amanaplanacanalpanama"

λ palindrome "amanaplanacanalpanama"

True
```

In **Get Programming with Haskell** I came across the following palindrome function:

```
isPalindrome :: String -> Bool
isPalindrome text = text == reverse text
```





While **isPalindrome** operates on **String**, which is a list of **Char**, **palindrome** is more generic in that it operates on a list of any type for which **==** is defined

```
palindrome :: Eq a => [a] -> Bool
palindrome = (≡) <*> reverse
```

But the signature of **isPalindrome** can also be made more generic, i.e. it can be changed to be the same as the signature of **palindrome**:

```
isPalindrome :: Eq a => [a] -> Bool
isPalindrome text = text == reverse text
```

And in fact, the illustration of **palindrome** by **Impure Pics** and **Ἑκάτη** points out that the definitions of **palindrome** and **isPalindrome** are equivalent by refactoring from one to the other.

```
palindrome = (≡) <*> reverse

palindrome a = (≡) a <*> reverse
             = (\x y -> x ≡ y) a (reverse
             = a ≡ reverse a
```

So, since (≡) is just (==), the only difference between **palindrome** and **isPalindrome** is that while the definition of **isPalindrome** is in **point-wise style** (it is **pointful**), the definition of **palindrome** is in **point-free style**.

@philip_schwarz

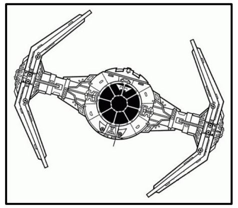
The palindrome function uses the <*> operator of the Function Applicative, which seems to me to be quite a niche Applicative instance.

To pave the way for an understanding of the <*> operator of the Function Applicative, let's first go through a refresher of the <*> operator of garden-variety applicatives like Maybe and List.

And before we do that, since every Applicative is also a Functor, (which is why Applicative is sometimes called Applicative Functor), let's gain an understanding of the map function of the Function Functor, but let's first warm up by going through a refresher of the map function of garden-variety functors like Maybe and List.

Tie fighter

<*> is called apply or ap, but is also known as the advanced tie fighter (|+| being the plain tie fighter), the angry parent, and the sad Pikachu.



advanced
Tie fighter

# Maybe Functor

Here is the definition of the **Functor type class**.

```haskell
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

```scala
trait Functor[F[_]]:
  def map[A,B](f: A => B): F[A] => F[B]
```

Yes, the more customary **Scala** signature for **map** is the one below. The above signature matches the **Haskell** one.

```scala
trait Functor[F[_]]:
  def map[A,B] (fa: F[A])(f: A => B): F[B]
```

And here is the definition of an **Algebraic Data Type** (**ADT**) modelling **optionality**.

```haskell
data  Maybe a  =  Nothing | Just a
```

```scala
enum Option[+A]:
  case Some(a: A)
  case None
```

Here is the definition of a **Functor instance** for the above **ADT**.

```haskell
instance Functor Maybe where
  fmap _ Nothing  = Nothing
  fmap f (Just a) = Just (f a)
```

```scala
given optionFunctor: Functor[Option] with
  override def map[A,B](f: A => B): Option[A] => Option[B] =
    case None => None
    case Some(a) => Some(f(a))
```

And here is an example of using the **map** function of the **Functor instance**.

```haskell
λ fmap reverse (Just "rab oof")
Just "foo bar"
λ fmap reverse Nothing
Nothing
```

```scala
def reverse(as: String): String = as.reverse

assert( map(reverse)(Some("rab oof")) == Some("foo bar"))
assert( map(reverse)(None) == None)
```

Now let's look at **List**, an **ADT** modelling **nondeterminism**. In **Haskell**, the syntax for lists is baked into the compiler, so below we show two examples of how the **List ADT** could look like if it were explicitly defined. In **Scala** there is no built-in syntax for lists, so the **List ADT** is defined explicitly. Below, we show a **List ADT** implemented using an **enum**.

**List Functor**

```
data List a = Nil | Cons a (List a)
```

```
data [a] = [ ] | a : [a]
```

```
enum List[+A]:
  case Cons(head: A, tail: List[A])
  case Nil
```

```
map :: (a -> b) -> [a] -> [b]
map _ []      = []
map f (x:xs) = f x : map f xs
```

The **map** function for **Haskell** lists, which is used below.

Here is the definition of a **Functor instance** for the above **ADT**.

```
instance Functor [] where
  fmap = map
```

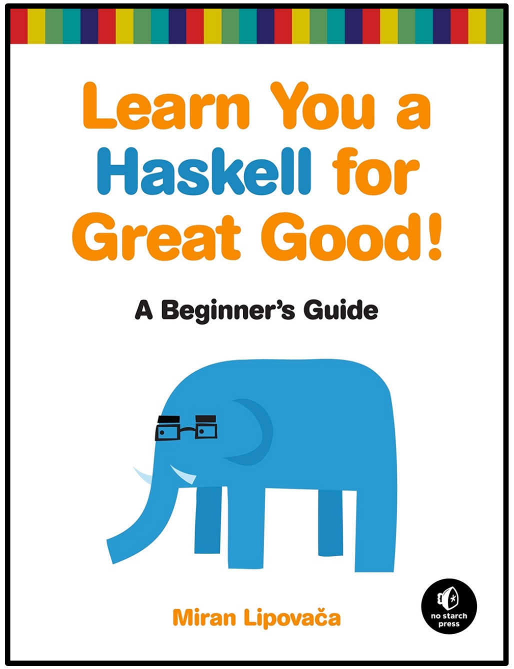
```
given listFunctor: Functor[List] with
  override def map[A, B](f: A => B): List[A] => List[B] =
    case Cons(a,as) => Cons(f(a),map(f)(as))
    case Nil => Nil
```

And here is an example of using the **map** function of the **Functor instance**.

```
λ fmap reverse ["rab oof"]
["foo bar"]
λ fmap reverse []
[]
```

```
def reverse(as: String): String = as.reverse

assert( map(reverse)(Cons("rab oof",Nil)) == Cons("foo bar",Nil))
assert( map(reverse)(Nil) == Nil)
```

While the plan was to look at the **map** function of **garden variety Functors** like **List** and **Option**, which it is possible to see as containers, let's also look at the **map** function of a **Functor** that cannot be seen as a container, i.e. **IO**. Here is how **Miran Lipovača** describes it.




Let's see how **IO** is an instance of **Functor**. When we **fmap** a function over an **I/O action**, we want to get back an **I/O action** that does the same thing but has our function applied over its result value. Here's the code:

```
instance Functor IO where
    fmap f action = do
        result <- action
        return (f result)
```

The result of mapping something over an **I/O action** will be an **I/O action**, so right off the bat, we use the **do** syntax to glue two **actions** and make a new one. In the implementation for **fmap**, we make a new **I/O action** that first performs the original **I/O action** and calls its result `result`. Then we do `return (f result)`. Recall that `return` is a function that makes an **I/O action** that doesn't do anything but only yields something as its result.

The action that a **do** block produces will always yield the result value of its last **action**. That's why we use `return` to make an **I/O action** that doesn't really do anything; it just yields `f result` as the result of the new **I/O action**.

Miran Lipovača

# IO Functor

```haskell
instance Functor IO where
    fmap f action = do
        result <- action
        return (f result)
```

```scala
final def map[B](f: A => B): IO[B] = {
  val trace = …
  Map(this, f, trace)
}
```

Cats Effect
The IO Monad for Cats

TYPELEVEL SCALA

```scala
final private[effect]
case class Pure[+A](a: A)
extends IO[A]

final private[effect]
case class Map[E, +A](source: IO[E], f: E => A, trace: AnyRef)
extends IO[A] with (E => IO[A]) {
  override def apply(value: E): IO[A] =
    Pure(f(value))
}
```

On the Scala side, we use the Cats Effect IO Monad, which being a Monad, is also a Functor.

If you are not so familiar with the IO Monad and you find some of this slide confusing then it's fine for you to just skip the slide.

```haskell
λ :type getLine
getLine :: IO String
```

```scala
import cats._
import cats.implicits._
import cats.effect.IO

def reverse(as: String): String = as.reverse
def getLine(): IO[String] = IO {scala.io.StdIn.readLine }

@main def main =
  val action = Functor[IO].map(getLine())(reverse)
  println(action.unsafeRunSync())

sbt run
rab oof
"foo bar"
```

Also on the Scala side, an IO value is an effectful value describing an I/O action which, when executed (by calling its unsafeRunSync method), results in an I/O side effect, which in this case is the reading of an input String from the console.

```haskell
λ fmap reverse getLine
rab oof
"foo bar"
```

@philip_schwarz

Now that we have warmed up by going through a refresher of the **map** function of functors for **Maybe**, **List** and **IO**, let’s turn to the **map** function of the **function Functor**.

## Functions As Functors

Another instance of **Functor** that we've been dealing with all along is `(->) r`. But wait! What the heck does `(->) r` mean? The function type `r -> a` can be rewritten as `(->) r a`, much like we can write `2 + 3` as `(+) 2 3`.

**When we look at it as `(->) r a`, we can see `(->)` in a slightly different light. It's just a type constructor that takes two type parameters, like Either**. But remember that a type constructor must take exactly one type parameter so it can be made an instance of **Functor**. That's why **we can't make `(->)` an instance of Functor; however, if we partially apply it to `(->) r`**, it doesn't pose any problems. If the syntax allowed for type constructors to be partially applied with sections (like we can partially apply + by doing (2+), which is the same as (+) 2), we could write (->) r as (r ->).

**How are functions functors? Let's take a look at the implementation**, which lies in Control.Monad.Instances:

```
instance Functor ((->) r) where
  fmap f g = (\x -> f (g x))
```

First, let's think about **fmap**'s type:

```
  fmap :: (a -> b) -> f a -> f b
```

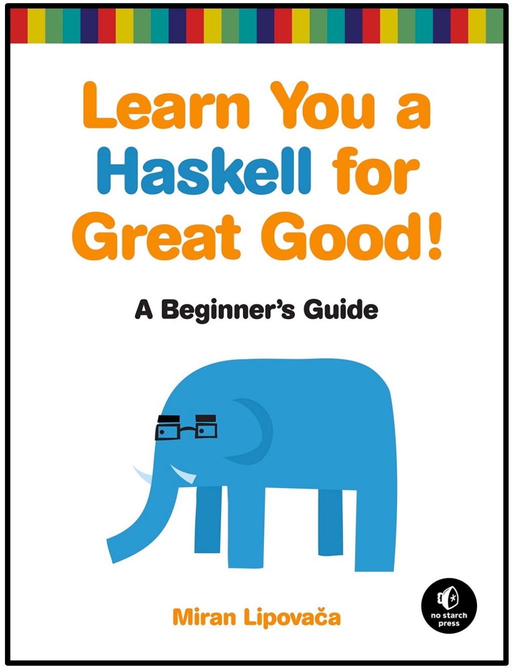
Next, let's mentally replace each **f**, which is the role that our functor instance plays, with `(->) r`. This will let us see how **fmap** should behave for this particular instance. Here's the result:

```
  fmap :: (a -> b) -> ((->) r a) -> ((->) r b)
```

Now we can write the `(->) r a` and `(->) r b` types as infix `r -> a` and `r -> b`, as we normally do with functions:

```
  fmap :: (a -> b) -> (r -> a) -> (r -> b)
```

Okay, **mapping a function over a function must produce a function**, just like mapping a function over a **Maybe** must produce a **Maybe**, and mapping a function over a list must produce a list. What does the preceding type tell us?




Miran Lipovača

```
fmap :: (a -> b) -> (r -> a) -> (r -> b)
```

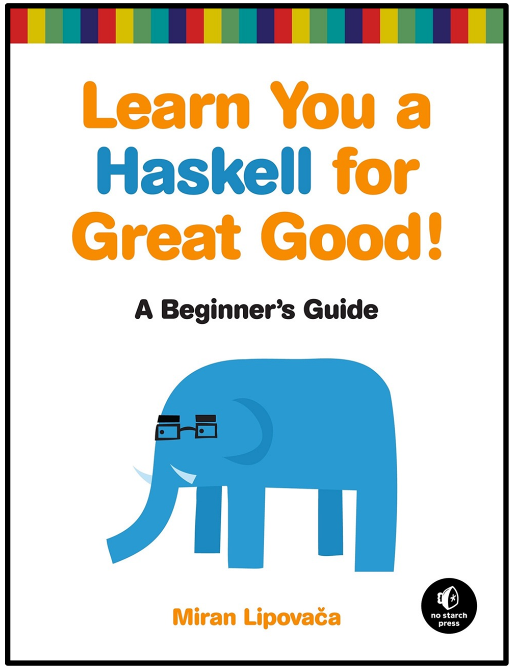
We see that it takes a function from a to b and a function from r to a and returns a function from r to b. Does this remind you of anything? Yes, **function composition**! We pipe the output of `r -> a` into the input of `(a -> b)` to get a function `r -> b`, which is exactly what **function composition** is all about. Here's another way to write this instance:

```
instance Functor ((->) r) where
  fmap = (.)
```

This makes it clear that **using fmap over functions is just function composition**. In a script,import Control.Monad.Instances, since that's where the instance is defined, and then load the script and try playing with mapping over functions:

```
ghci> :t fmap (*3) (+100)
fmap (*3) (+100) :: (Num a) => a -> a

ghci> fmap (*3) (+100) 1
303

ghci> (*3) `fmap` (+100) $ 1
303

ghci> (*3) . (+100) $ 1
303

ghci> fmap (show . (*3)) (+100) 1
"303"
```

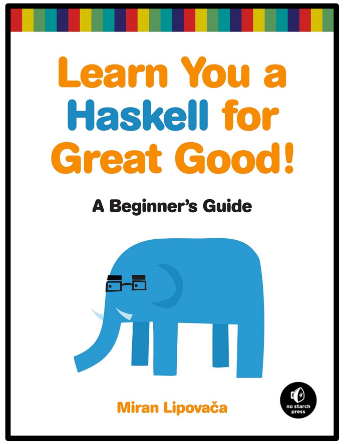
We can call **fmap** as an infix function so that the resemblance to . is clear. **In the second input line, we're mapping** (*3) **over** (+100)**, which results in a function that will take an input, apply** (+100) **to that, and then apply** (*3) **to that result. We then apply that function to** 1.




Miran Lipovača

Just like all functors, functions can be thought of as values with contexts. When we have a function like (+3), we can view the value as the eventual result of the function, and the context is that we need to apply the function to something to get to the result. Using fmap (*3) on (+100) will create another function that acts like (+100), but before producing a result, (*3) will be applied to that result.

The fact that fmap is function composition when used on functions isn't so terribly useful right now, but at least it's very interesting. It also bends our minds a bit and lets us see how things that act more like computations than boxes (IO and (->) r) can be functors. The function being mapped over a computation results in the same sort of computation, but the result of that computation is modified with the function.




I think that the use of partially applied functions like (*3) and (+100) in the examples on the previous slide could be making the examples slightly harder to understand, by adding some unnecessary complexity. So here are the examples again but this time using single-argument functions twice and square.

Using fmap to compose the twice function with the square function

```
λ fmap twice square 3
18

λ twice `fmap` square $ 3
18

λ twice . square $ 3
18

λ fmap (show . twice) square 3
"18"
```

```
λ twice n = n + n
λ square n = n * n

λ :type twice
twice :: Num a => a -> a

λ :type square
square :: Num a => a -> a

λ :type fmap twice square
fmap twice square :: Num b => b -> b
```

The $ operator allows us to write twice `fmap` square $ 3 rather than ( twice `fmap` square) 3

See an upcoming slide for a fuller explanation of the $ operator.

Earlier on we saw **Functor** instances for **Maybe**, **List** and **IO**, plus an example of their usage. So here is a **Functor** instance for **functions** and an example of its usage.

# Function Functor

```haskell
instance Functor ((->) r) where
  fmap = (.)
```

```scala
given functionFunctor[D]: Functor[[C] =>> D => C] with
  override def map[A,B](f: A => B): (D => A) => (D => B) =
    g => f compose g
```

In **Haskell**, we are using type variable **r** as the domain of the functions supported by the **Functor** instance.

To achieve a similar effect in **Scala**, we are using **type lambda** `[C] =>> D => C`. `D` is the domain of a function and C is its codomain. So `functionFunctor[Int]` for example, is the functor instance for functions whose domain is `Int`. i.e. functions of type Int => C for any C.

```haskell
λ :type reverse
reverse :: [a] -> [a]

λ :type words
words :: String -> [String]

λ words "one two"
["one","two"]
```

```scala
def reverse[A]: List[A] => List[A] =
  _.reverse

def words: String => List[String] =
  s => s.split(" ").toList

assert( words("one two") == List("one", "two") )
```

```haskell
λ fmap reverse words "rab oof"
["oof","rab"]
```

```scala
val stringFunctionFunctor = functionFunctor[String]
import stringFunctionFunctor.map

assert( map(reverse)(words)("rab oof") == List("oof", "rab"))
```

Here is a quick recap of the behaviours of the **fmap** functions of the four **Functor** instances we looked at.

```
reversed :: Maybe String
reversed = fmap reverse (Just "rab oof")

λ reversed
Just "foo bar"
```

```
reversed :: Maybe String
reversed = fmap reverse Nothing

λ reversed
Nothing
```

Mapping a function **f** over **Nothing** yields **Nothing**.

```
reversed :: [String]
reversed = fmap reverse ["rab oof"]

λ reversed
["foo bar"]
```

```
reversed :: [String]
reversed = fmap reverse []

λ reversed
[]
```

Mapping a function **f** over an **empty list** yields an **empty list**.

```
reversed :: IO String
reversed = fmap reverse getLine

λ reversed
rab oof
"foo bar"
```

```
reversed :: IO String
reversed = fmap reverse getLine

λ reversed
<no one types anything, so the
function hangs waiting for input>
```

Mapping a function **f** over an **IO action** that ends up producing an unwanted **side effect**, results in an **action** producing the same **side effect**.

```
reversed :: String -> [String]
reversed = fmap reverse words

λ reversed "rab oof"
["oof","rab"]
```

```
reversed :: a -> [a]
reversed = fmap reverse id

λ reversed "rab oof"
"foo bar"
```

Mapping a function **f** over the **identity** function, i.e. the function that does nothing, simply yields **f**.

```
reverse :: [a] -> [a]
```

```
words :: String -> [String]
```

A couple of slides ago we came across the $ operator. Here is its definition.

**function application**
```
($) :: (a -> b) -> a -> b
f $ x =  f x
```

Application operator. This operator is redundant, since ordinary application (f x) means the same as (f $ x). However, $ has low, right-associative binding precedence, so it sometimes allows parentheses to be omitted; for example:

```
f $ g $ h x = f (g (h x))
```

And below is the definition of <$>, an operator that is closely related to $.

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

**function application lifted** over a **Functor**.
```
(<$>) :: Functor f => (a -> b) -> f a -> f b
(<$>) = fmap
```

An infix synonym for **fmap**.

The name of this operator is an allusion to $. Note the similarities between their types:

```
($) :: (a -> b) -> a -> b
(<$>) :: Functor f => (a -> b) -> f a -> f b
```

Whereas $ is **function application**, <$> is **function application lifted** over a **Functor**.

Here is how our usages of **fmap** look when we switch to **<$>**

```
fmap reverse (Just "rab oof")
```

```
fmap reverse ["rab oof"]
```

```
fmap reverse getLine
```

```
fmap reverse words
```

```
reverse <$> (Just "rab oof")
```

```
reverse <$> ["rab oof"]
```

```
reverse <$> getLine
```

```
reverse <$> words
```

**<$>** will prove much more useful in future slides in which we look at **Applicative**.

For what it is worth, here we just add <$> to the Scala definition of Functor, and have a quick go at switching from map to <$> in an example using the Int function Functor.

```
trait Functor[F[_]]:
  def map[A,B](f: A => B): F[A] => F[B]
  extension[A,B] (f: A => B)
    def `<$>`(fa: F[A]): F[B] = map(f)(fa)

given functionFunctor[D]: Functor[[C] =>> D => C] with
  override def map[A,B](f: A => B): (D => A) => (D => B) =
    g => f compose g

val intFunctionFunctor = functionFunctor[Int]
import intFunctionFunctor.map

val twice: Int => Int = x => x + x
val square: Int => Int = x => x * x

assert(map(twice)(square)(5) == 50)
assert((twice `<$>` square)(5) == 50)
```

Again, <$> will become more useful when we look at Applicative.

@philip_schwarz

The palindrome function uses the <*> operator of the Function Applicative, so after gaining an understanding of the map function of the Function Functor, and in order to pave the way for an understanding of the <*> operator of the Function Applicative, let’s first go through a refresher of the <*> operator of garden-variety applicatives like Maybe and List.

Here is the definition of the **Applicative type class**.

```haskell
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

```scala
trait Functor[F[_]]:

  def map[A,B](f: A => B): F[A] => F[B]

  extension[A,B] (f: A => B)
    def `<$>`(fa: F[A]): F[B] = map(f)(fa)
```

```haskell
class Functor f => Applicative f where
   pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

```scala
trait Applicative[F[_]] extends Functor[F]:

  def pure[A](a: => A): F[A]

  def apply[A,B](fab: F[A => B]): F[A] => F[B]

  extension[A,B] (fab: F[A => B])
    def <*> (fa: F[A]): F[B] = apply(fab)(fa)
```

## Maybe Applicative

And here is an Applicative instance for the Maybe ADT.

```haskell
data  Maybe a  =  Nothing | Just a
```

```scala
enum Option[+A]:
  case Some(a: A)
  case None
```

```haskell
instance Functor Maybe where
fmap _ Nothing  = Nothing
fmap f (Just a) = Just (f a)
```

```haskell
instance Applicative Maybe where
pure = Just
Nothing <*> _ = Nothing
Just f  <*> m = fmap f m
```

```scala
given optionApplicative as Applicative[Option]:
  override def map[A, B](f: A => B): Option[A] => Option[B] =
    case None => None
    case Some(a) => Some(f(a))
  override def pure[A](a: => A): Option[A] = Some(a)
  override def apply[A,B](fab: Option[A => B]): Option[A] => Option[B] =
    fa => (fab, fa) match
      case (None,_) => None
      case (Some(f),m) => map(f)(m)
```

See below for some examples of using the Maybe Applicative to apply a function that takes a single argument.

```haskell
twice n = n + n
```

```scala
val twice: Int => Int = n => n + n
```

```haskell
λ Nothing <*> Just 2
Nothing

λ Just twice <*> Nothing
Nothing

λ Nothing <*> Nothing
Nothing

λ Just twice <*> Just 2
Just 4

λ pure twice <*> Just 2
Just 4

λ fmap twice (Just 2)
Just 4

λ twice <$> Just 2
Just 4
```

```scala
assert( None <*> Some(2) == None )

assert( Some(twice) <*> None == None )

assert(None <*> None == None )

assert( Some(twice) <*> Some(2) == Some(4) )

assert( pure(twice) <*> Some(2) == Some(4) )

assert( map(twice)(Some(2)) == Some(4) )

assert( twice `<$>` Some(2) == Some(4) )
```

And now some examples of using the **Maybe Applicative** to apply a function that takes two arguments.

```haskell
(+) :: Num a => a -> a -> a
```

```scala
val `(+)`: Int => Int => Int = x => y => x + y
```

```haskell
λ Nothing <*> Just 2 <*> Just 3
Nothing

λ Just (+) <*> Nothing <*> Just 3
Nothing

λ Nothing <*> Nothing <*> Just 3
Nothing

λ Just (+) <*> Just 2 <*> Just 3
Just 5

λ pure (+) <*> Just 2 <*> Just 3
Just 5

λ fmap (+) (Just 2) <*> Just 3
Just 5

λ (+) <$> Just 2 <*> Just 3
Just 5
```

```scala
assert( None <*> Some(2) <*> Some(3) == None )

assert( Some(`(+)`) <*> None <*> Some(3) == None )

assert( None <*> None <*> Some(3) == None )

assert( Some(`(+)`) <*> Some(2) <*> Some(3) == Some(5) )

assert( pure(`(+)`) <*> Some(2) <*> Some(3) == Some(5) )

assert( map(`(+)`)(Some(2)) <*> Some(3) == Some(5) )

assert( `(+)` `<$>` Some(2) <*> Some(3) == Some(5) )
```

@philip_schwarz

At the bottom of the previous slide we see that

```
pure (+) <*> Just 2 <*> Just 3
```

is equivalent to

```
fmap (+) (Just 2) <*> Just 3
```

which in turn is equivalent to

```
(+) <$> Just 2 <*> Just 3
```

i.e. by using `<$>` together with `<*>` we can write the invocation of a function with arguments that are in an **applicative** context **m**

```
f <$> mx <*> my <*> mz
```

in a way that is very similar to the invocation of the function with ordinary arguments

```
f x y z
```

See the next slide for how **Miran Lipovača** puts it .

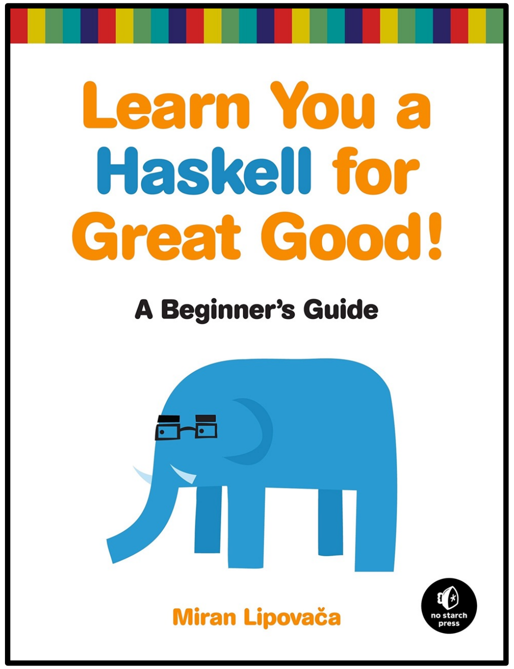
## The Applicative Style

With the Applicative type class, we can chain the use of the <*> function, thus enabling us to seamlessly operate on several applicative values instead of just one. For instance, check this out:

```
ghci> pure (+) <*> Just 3 <*> Just 5
Just 8
ghci> pure (+) <*> Just 3 <*> Nothing
Nothing
ghci> pure (+) <*> Nothing <*> Just 5
Nothing
```

We wrapped the + function inside an applicative value and then used <*> to call it with two parameters, both applicative values … Isn't this awesome? Applicative functors and the applicative style of pure f <*> x <*> y <*> … allow us to take a function that expects parameters that aren't applicative values and use that function to operate on several applicative values. The function can take as many parameters as we want, because it's always partially applied step by step between occurrences of <*>.

This becomes even more handy and apparent if we consider the fact that pure f <*> x equals fmap f x. This is one of the applicative laws… pure puts a value in a default context. If we just put a function in a default context and then extract and apply it to a value inside another applicative functor, that's the same as just mapping that function over that applicative functor. Instead of writing pure f <*> x <*> y <*> ..., we can write fmap f x <*> y <*> ....



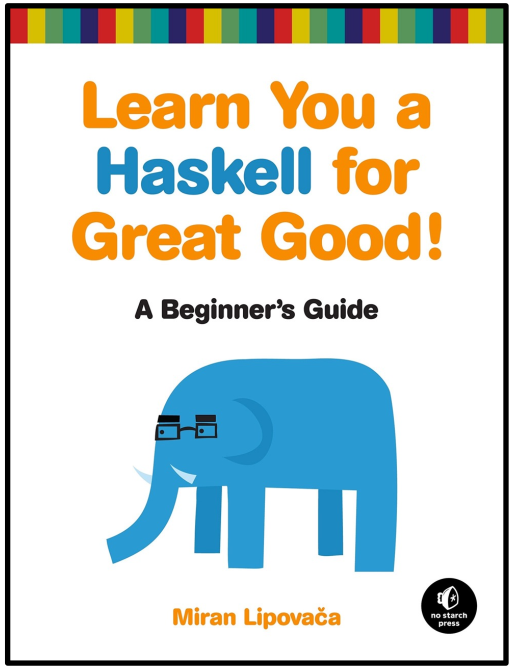

Miran Lipovača

**This is why `Control.Applicative` exports a function called <$>, which is just fmap as an infix operator. Here's how it's defined:**

```
(<$>) :: (Functor f) => (a -> b) -> f a -> f b
f <$> x = fmap f x
```

NOTE: Remember that type variables are independent of parameter names or other value names. The **f** in the function declaration here is a type variable with a class constraint saying that any type constructor that replaces **f** should be in the **Functor** type class. The **f** in the function body denotes a function that we map over **x**. The fact that we used **f** to represent both of those doesn't mean that they represent the same thing.

**By using <$>, the applicative style really shines, because now if we want to apply a function** f **between three applicative values, we can write** f <$> x <*> y <*> z. **If the parameters were normal values rather than applicative functors, we would write** f x y z.




Miran Lipovača

What about an Applicative instance for the List ADT? Here is how it looks in Haskell, followed by some examples of its usage.

```haskell
instance Applicative [] where
pure x = [x]
fs <*> xs = [f x | f <- fs, x <- xs]
```

```haskell
class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

```
λ :type (,)
(,) :: a -> b -> (a, b)

λ (,) <$> ['a','b'] <*> [1,2]
[('a',1),('a',2),('b',1),('b',2)]
```

```
λ (+) <$> [1,2,3] <*> [10,20,30]
[11,21,31,12,22,32,13,23,33]

λ (+) <$> [1,2,3] <*> [10]
[11,12,13]

λ (+) <$> [1,2,3] <*> []
[]
```

```
λ max3 x y z = max x (max y z)

λ max3 <$> [6,2] <*> [3,5] <*> [4,9]
[6,9,6,9,4,9,5,9]

λ max3 <$> [6,2] <*> [3] <*> [4,9]
[6,9,4,9]

λ max3 <$> [6,2] <*> [] <*> [4,9]
[]

λ max3 <$> [] <*> [3,5] <*> [4,9]
[]
```

```
λ [inc, twice, square] <*> [1,2,3]
[2,3,4,2,4,6,1,4,9]

λ [inc, twice, square] <*> [3]
[4,6,9]

λ [inc, twice, square] <*> []
[]

λ [inc] <*> [1,2,3]
[2,3,4]

λ [] <*> [1,2,3]
[]
```

```
λ inc n = n + 1
λ twice n = n + n
λ square n = n * n
```

```
λ [(+),(*)] <*> [10,20,30] <*> [1,2]
[11,12,21,22,31,32,10,20,20,40,30,60]

λ [(+),(*)] <*> [10,20,30] <*> [2]
[12,22,32,20,40,60]

λ [(+),(*)] <*> [10] <*> [1,2]
[11,12,10,20]

λ [(+),(*)] <*> [] <*> [1,2]
[]

λ [(+),(*)] <*> [10,20,30] <*> []
[]

λ [(+)] <*> [10,20,30] <*> [1,2]
[11,12,21,22,31,32]

λ [] <*> [10,20,30] <*> [1,2]
[]
```

So what does the Scala implementation of the Applicative instance for the List ADT look like?

```
instance Applicative [] where
pure x = [x]
fs <*> xs = [f x | f <- fs, x <- xs]
```

Implementing pure in Scala is trivial:

```
def pure[A](a: => A): List[A] = Cons(a, Nil)
```

```
enum List[+A]:
  case Cons(head: A, tail: List[A])
  case Nil
```

As for <*>, the above definition uses a list comprehension, firstly to extract function f from the list which is its first parameter, and secondly to extract the function's argument x from the list which is its second parameter. It then applies f to x and returns a singleton list containing the result.

How do we do the equivalent in Scala, where there is no list comprehension? For type constructors that are Monads, i.e. they have both a map method and a flatMap method, Scala provides for comprehensions. E.g. if we were implementing the List Monad, we would be able to implement Applicative's apply method as follows:

```
override def apply[A,B](fab: List[A => B]): List[A] => List[B] = fa =>
  for {
    f <- fab
    a <- fa
  } yield f(a)
```

But since in our particular case we are implementing the List Applicative rather than the List Monad, we have to find another way of implementing the apply method.

If we define an **append** function for **List**, then we can define the **apply** function **recursively** as follows:

```
override def apply[A,B](fab: List[A => B]): List[A] => List[B] = fa =>
  (fab, fa) match
    case (Nil, _) => Nil
    case (Cons(f,Nil), Nil) => Nil
    case (Cons(f,Nil), Cons(a,as)) => Cons(f(a),apply(fab)(as))
    case (Cons(f,fs), fa) => append( apply(Cons(f, Nil))(fa), apply(fs)(fa))
```

Let’s define **append** in terms of the ubiquitous **foldRight** function:

```
object List:
  def append[A](lhs: List[A], rhs: List[A]): List[A] =
    foldRight((h: A,t: List[A]) => Cons(h,t))(rhs)(lhs)
  def foldRight[A,B](f: (A, B) => B)(b: B)(as: List[A]): B =
    as match
      case Nil => b
      case Cons(h,t) => f(h,foldRight(f)(b)(t))
```

We can also use **foldRight** to implement the **map** function:

```
def map[A, B](f: A => B): List[A] => List[B] =
  fa => foldRight((h:A,t:List[B]) => Cons(f(h),t))(Nil)(fa)
```

See the next slide for the final code for this approach to the **List Applicative**, including a small test example.

```scala
enum List[+A]:
  case Cons(head: A, tail: List[A])
  case Nil
```

# List Applicative

```scala
given listApplicative: Applicative[List] with
  override def pure[A](a: => A): List[A] = Cons(a, Nil)
  override def map[A, B](f: A => B): List[A] => List[B] =
    fa => foldRight((h:A,t:List[B]) => Cons(f(h),t))(Nil)(fa)
  override def apply[A,B](fab: List[A => B]): List[A] => List[B] = fa =>
    (fab, fa) match
      case (Nil, _) => Nil
      case (Cons(f,Nil), Nil) => Nil
      case (Cons(f,Nil), Cons(a,as)) => Cons(f(a),apply(fab)(as))
      case (Cons(f,fs), fa) => append( apply(Cons(f, Nil))(fa), apply(fs)(fa))
```

```scala
trait Functor[F[_]]:
  def map[A,B](f: A => B): F[A] => F[B]
  extension[A,B] (f: A => B)
    def `<$>`(fa: F[A]): F[B] = map(f)(fa)
```

```scala
trait Applicative[F[_]] extends Functor[F]:
  def pure[A](a: => A): F[A]
  def apply[A,B](fab: F[A => B]): F[A] => F[B]
  extension[A,B] (fab: F[A => B])
    def <*> (fa: F[A]): F[B] = apply(fab)(fa)
```

```scala
object List:
  def append[A](lhs: List[A], rhs: List[A]): List[A] =
    foldRight((h: A,t: List[A]) => Cons(h,t))(rhs)(lhs)
  def foldRight[A,B](f: (A, B) => B)(b: B)(as: List[A]): B = as match
    case Nil => b
    case Cons(h,t) => f(h,foldRight(f)(b)(t))
  def of[A](as: A*): List[A] = as match
    case Seq() => Nil
    case _ => Cons(as.head, of(as.tail: _*))
```

```scala
val inc: Int => Int = x => x + 1
val twice: Int => Int = x => x + x
val square: Int => Int = x => x * x

assert( List.of(inc, twice, square) <*> List.of(1, 2, 3) == List.of(2, 3, 4, 2, 4, 6, 1, 4, 9) )
assert( List.of(inc, twice, square) <*> Nil == Nil )
assert( List.of(inc, twice, square) <*> List.of(3) == List.of(4,6,9) )
assert( List.of(inc) <*> List.of(1,2,3) == List.of(2,3,4) )
assert( Nil <*> List.of(1, 2, 3) == Nil )
assert( Nil <*> Nil == Nil )
```

Each function in the first list gets applied to each argument in the second list.

@philip_schwarz

Before we move on, I'd like to share another way of implementing the List Applicative which is very neat, but which turns out to be cheating, in that it amounts to making the List a Monad. If we take the code on the previous slide, then all we have to do is give List a flatten function:

```
def flatten[A](xss: List[List[A]]): List[A] =
  foldRight[List[A],List[A]](append)(Nil)(xss)
```

This is cheating, because giving List the functions pure, map and flatten, is equivalent to giving it functions pure and flatMap, i.e. making it a Monad.

We can now greatly simplify the apply function as follows:

```
def apply[A,B](fab: List[A => B]): List[A] => List[B] = fa =>
  flatten(
    map((f: A => B) =>
      map(a => f(a))(fa))(fab))
```

Another interesting thing is that if we also give the Maybe Applicative a fold function, then we can give it a flatten function…

```
enum Option[+A]:
  case Some(a: A)
  case None
  def fold[B](ifEmpty: B)(f: A => B): B = this match
    case Some(a) => f(a)
    case None => ifEmpty
```

```
object Option:
  def flatten[A](ooa: Option[Option[A]]): Option[A] =
    ooa.fold(None)(identity)
```

…which means we can define the apply function the same way we did for List.

```
def apply[A,B](fab: Option[A => B]): Option[A] => Option[B] = fa =>
  flatten(
    map((f: A => B) =>
      map(a => f(a))(fa))(fab))
```

By the way: if we did the above, we could also define map in terms of fold.

```
def map[A, B](f: A => B): Option[A] => Option[B] =
  fa => fa.fold(None)(a => Some(f(a)))
```

The next slide just shows the Scala code for the List and Option Applicative instances after applying the approach described in the previous slide.

```scala
enum List[+A]:
  case Cons(head: A, tail: List[A])
  case Nil
```

```scala
enum Option[+A]:
  case Some(a: A)
  case None
  def fold[B](ifEmpty: B)(f: A => B): B = this match
    case Some(a) => f(a)
    case None => ifEmpty
```

```scala
given listApplicative: Applicative[List] with
  override def pure[A](a: => A): List[A] = Cons(a, Nil)
  override def map[A, B](f: A => B): List[A] => List[B] =
    fa => foldRight((h:A,t:List[B]) => Cons(f(h),t))(Nil)(fa)
  override def apply[A,B](fab: List[A => B]): List[A] => List[B] =
    fa =>
      flatten(
        map((f: A => B) =>
          map(a => f(a))(fa))(fab))
```

```scala
given optionApplicative: Applicative[Option] with
  override def pure[A](a: => A): Option[A] = Some(a)
  override def map[A, B](f: A => B): Option[A] => Option[B] =
    fa => fa.fold(None)(a => Some(f(a)))
  override def apply[A,B](fab: Option[A=>B]): Option[A] => Option[B] =
    fa =>
      flatten(
        map((f: A => B) =>
          map(a => f(a))(fa))(fab))
```

```scala
object List:
  def flatten[A](xss: List[List[A]]): List[A] =
    foldRight[List[A],List[A]](append)(Nil)(xss)
  def append[A](lhs: List[A], rhs: List[A]): List[A] =
    foldRight((h: A,t: List[A]) => Cons(h,t))(rhs)(lhs)
  def foldRight[A,B](f: (A, B) => B)(b: B)(as: List[A]): B =
    as match
      case Nil => b
      case Cons(h,t) => f(h,foldRight(f)(b)(t))
```

```scala
object Option:
  def flatten[A](ooa: Option[Option[A]]): Option[A] =
    ooa.fold(None)(identity)
```

```scala
trait Functor[F[_]]:
  def map[A,B](f: A => B): F[A] => F[B]
  extension[A,B] (f: A => B)
    def `<$>`(fa: F[A]): F[B] = map(f)(fa)
```

```scala
trait Applicative[F[_]] extends Functor[F]:
  def pure[A](a: => A): F[A]
  def apply[A,B](fab: F[A => B]): F[A] => F[B]
  extension[A,B] (fab: F[A => B])
    def <*> (fa: F[A]): F[B] = apply(fab)(fa)
```

In the next 3 slides we take a look at the **IO Applicative**.

## IO Is An Applicative Functor, Too

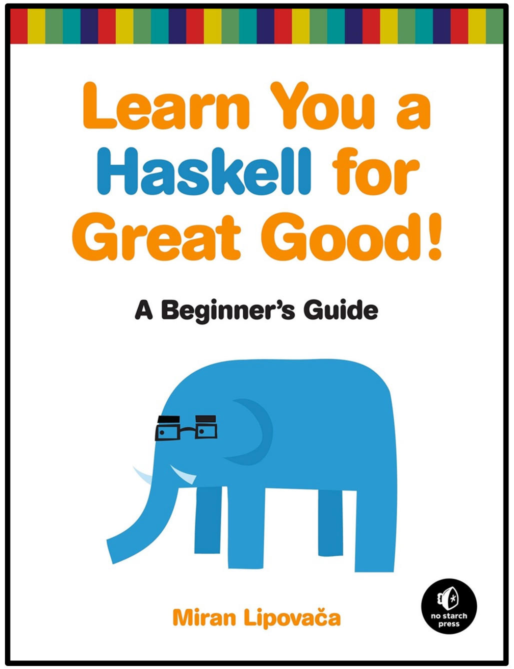
Another instance of **Applicative** that we've already encountered is **IO**. This is how the instance is implemented:

```
instance Applicative IO where
  pure = return
  a <*> b = do
    f <- a
    x <- b
    return (f x)
```

Since `pure` is all about putting a value in a minimal context that still holds the value as the result, it makes sense that `pure` is just `return`. return makes an **I/O action** that doesn't do anything. It just yields some value as its result, without performing any **I/O operations** like printing to the terminal or reading from a file.

If `<*>` were specialized for `IO`, it would have a type of `(<*>) :: IO (a -> b) -> IO a -> IO b`. In the case of `IO`, it takes the **I/O action a**, which yields a function, performs the function, and binds that function to **f**. Then it performs **b** and binds its result to **x**. Finally, it applies the function **f** to **x** and yields that as the result. We used **do** syntax to implement it here. (Remember that do syntax is about taking several **I/O actions** and gluing them into one.)

With `Maybe` and `[]`, we could think of `<*>` as simply extracting a function from its left parameter and then applying it over the right one. With `IO`, extracting is still in the game, but now we also have a notion of ***sequencing***, because we're taking two **I/O actions** and gluing them into one. We need to extract the function from the first **I/O action**, but to extract a result from an **I/O action**, it must be performed.



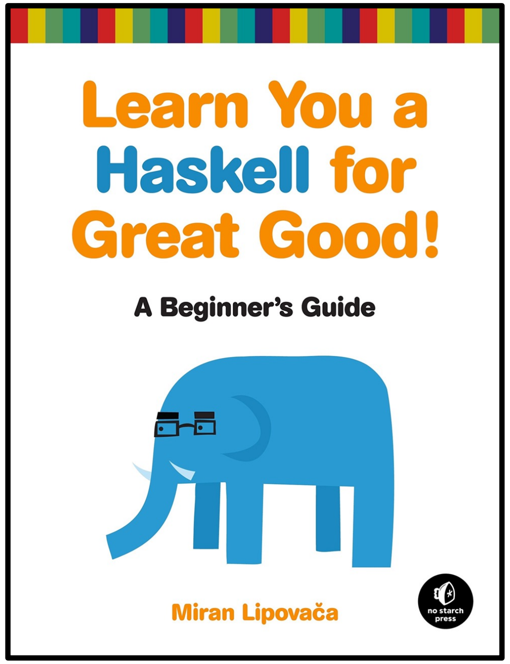

Miran Lipovača

Consider this:

```
myAction :: IO String
myAction = do
  a <- getLine
  b <- getLine
  return $ a ++ b
```

This is an **I/O action** that will prompt the user for two lines and yield as its result those two lines concatenated. We achieved it by gluing together two `getLine` **I/O actions** and a **`return`**, because we wanted our new glued **I/O action** to hold the result of `a ++ b`. Another way of writing this is to use the **applicative style**:

```
myAction :: IO String
myAction = (++) <$> getLine <*> getLine
```




Let's try out the IO Applicative.

```
instance Applicative IO where
    pure = return
    a <*> b = do
      f <- a
      x <- b
      return (f x)
```

```
λ :type getLine
getLine :: IO String
```

```
λ pure (++) <*> getLine <*> getLine
foo
bar
"foobar"
```

```
λ (++) <$> getLine <*> getLine
foo
bar
"foobar"
```

Here we try out the IO Applicative in Scala using the Cats Effect IO Monad, which being a Monad, is also an Applicative Functor.

@philip_schwarz

```
import cats._
import cats.implicits._
import cats.effect.IO

extension (l: String)
  def `(++)`(r: String): String = l ++ r

extension[A,B,F[_]] (f: A => B)
  def `<$>`(fa: F[A])(using functor: Functor[F]): F[B] = functor.map(fa)(f)

def getLine(): IO[String] = IO.pure(scala.io.StdIn.readLine)

@main def main =

  val action1: IO[String] = IO.pure(`(++)`) <*> getLine() <*> getLine()
  println(action1.unsafeRunSync)

  val action2: IO[String] = `(++)` `<$>` getLine() <*> getLine()
  println(action2.unsafeRunSync)

sbt run
foo
bar
"foobar"
abc
def
"abcdef"
```

# IO Applicative

Cats Effect
The IO Monad for Cats

TYPELEVEL SCALA

Now that we have looked at the **Applicative** instances for **Maybe**, for **lists**, and for **IO** actions, let's see how **Graham Hutton** describes **applicative style** and summarises the similarities and differences between the three types of programming supported by the **applicative style** for the three above instances of **Applicative**.

```haskell
pure :: a -> f a
(<*>) :: f (a -> b) -> f a -> f b
```

That is, **pure** converts a value of type **a** into a structure of type **f a**, while **<*> is a generalised form of function application for which the argument function, the argument value, and the result value are all contained in f structures**. As with **normal function application**, the **<*>** operator is written between its two arguments and is assumed to associate to the left. For example,

```haskell
g <*> x <*> y <*> z
```

means

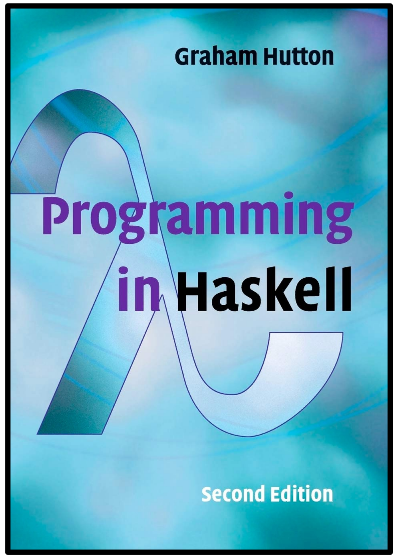
```haskell
((g <*> x) <*> y) <*> z
```

A typical use of pure and **<*>** has the following form:

```haskell
pure g <*> x1 <*> x2 <*> ... <*> xn
```

Such expressions are said to be in **applicative style**, because of the similarity to **normal function application** notation `g x1 x2 ... xn`.

In both cases, **g** is a **curried function** that takes n arguments of type **a1 ... an** and produces a result of type **b**. However, in **applicative style**, each argument **xi** has type **f ai** rather than just **ai**, and the overall result has type **f b** rather than **b**.




Graham Hutton
@haskellhutt

the applicative style for Maybe supports a form of exceptional programming in which we can apply pure functions to arguments that may fail without the need to manage the propagation of failure ourselves, as this is taken care of automatically by the applicative machinery.

…

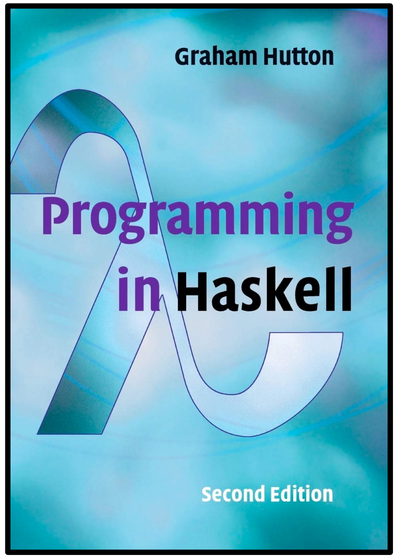
the applicative style for lists supports a form of non-deterministic programming in which we can apply pure functions to multi-valued arguments without the need to manage the selection of values or the propagation of failure, as this is taken care of by the applicative machinery.

…

the applicative style for IO supports a form of interactive programming in which we can apply pure functions to impure arguments without the need to manage the sequencing of actions or the extraction of result values, as this is taken care of automatically by the applicative machinery.

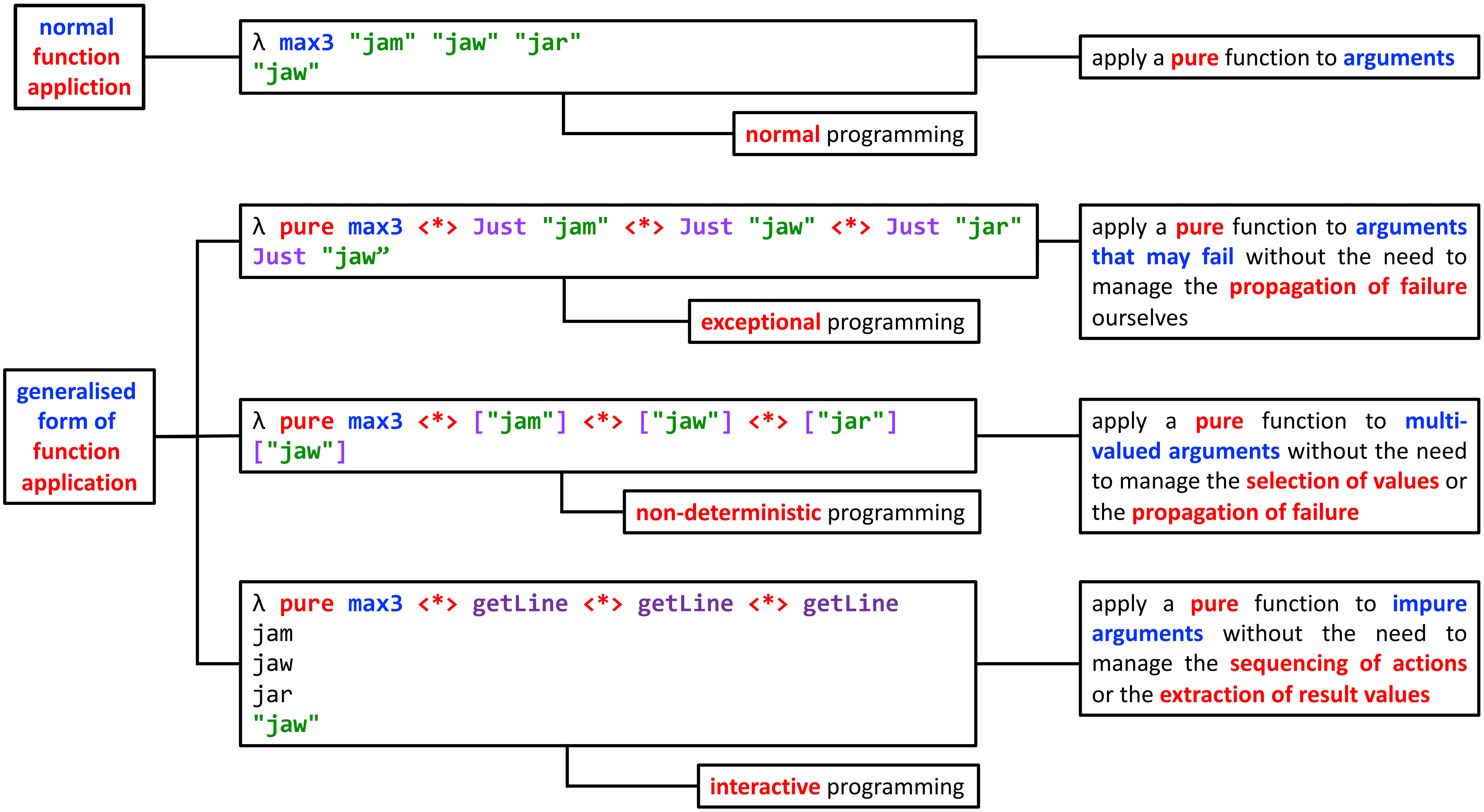
…

The common theme between the instances is that they all concern programming with effects. In each case, the applicative machinery provides an operator <*> that allows us to write programs in a familiar applicative style in which functions are applied to arguments, with one key difference: the arguments are no longer just plain values but may also have effects, such as the possibility of failure, having many ways to succeed, or performing input/output actions. In this manner, applicative functors can also be viewed as abstracting the idea of applying pure functions to effectful arguments, with the precise form of effects that are permitted depending on the nature of the underlying functor.




Graham Hutton
@haskellhutt

The next slide is a diagram illustrating the above with an example


normal function appliction
λ max3 "jam" "jaw" "jar"
"jaw"
apply a pure function to arguments
normal programming
generalised form of function application
λ pure max3 <*> Just "jam" <*> Just "jaw" <*> Just "jar"
Just "jaw”
apply a pure function to arguments that may fail without the need to manage the propagation of failure ourselves
exceptional programming
λ pure max3 <*> ["jam"] <*> ["jaw"] <*> ["jar"]
["jaw"]
apply a pure function to multi-valued arguments without the need to manage the selection of values or the propagation of failure
non-deterministic programming
λ pure max3 <*> getLine <*> getLine <*> getLine
jam
jaw
jar
"jaw"
apply a pure function to impure arguments without the need to manage the sequencing of actions or the extraction of result values
interactive programming

Same code as on the previous slide, but this time using infix map .operator <$> .

```
λ max3 "jam" "jaw" "jar"
"jaw"
```

```
λ max3 <$> Just "jam" <*> Just "jaw" <*> Just "jar"
Just "jaw”
```

```
λ max3 <$> ["jam"] <*> ["jaw"] <*> ["jar"]
["jaw"]
```

```
λ max3 <$> getLine <*> getLine <*> getLine
jam
jaw
jar
"jaw"
```

@philip_schwarz

Now let's turn to the **Applicative** instance for **functions**. In the next two slides we look at how **Miran Lipovača** describes it.

## Functions As Applicatives

Another instance of Applicative is `(->) r`, or functions. We don't often use functions as applicatives, but the concept is still really interesting, so let's take a look at how the function instance is implemented.

```
instance Applicative ((->) r) where
pure x = (\_ -> x)
f <*> g = \x -> f x (g x)
```

When we wrap a value into an applicative value with pure, the result it yields must be that value. A minimal default context still yields that value as a result. That's why in the function instance implementation, pure takes a value and creates a function that ignores its parameter and always returns that value. The type for pure specialized for the `(->) r` instance is `pure :: a -> (r -> a)`.

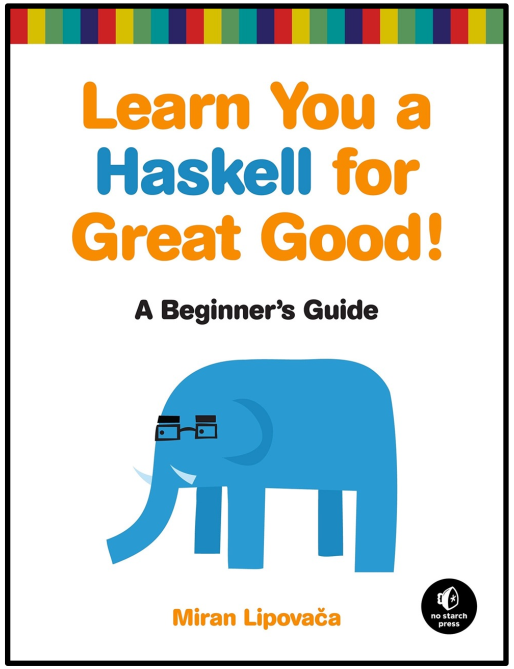
```
ghci> (pure 3) "blah"
3
```

Because of currying, function application is left-associative, so we can omit the parentheses.

```
ghci> pure 3 "blah"
3
```

The instance implementation for `<*>` is a bit cryptic, so let's just take a look at how to use functions as applicative functors in the applicative style:

```
ghci> :t (+) <$> (+3) <*> (*100)
(+) <$> (+3) <*> (*100) :: (Num a) => a -> a
```




Miran Lipovača

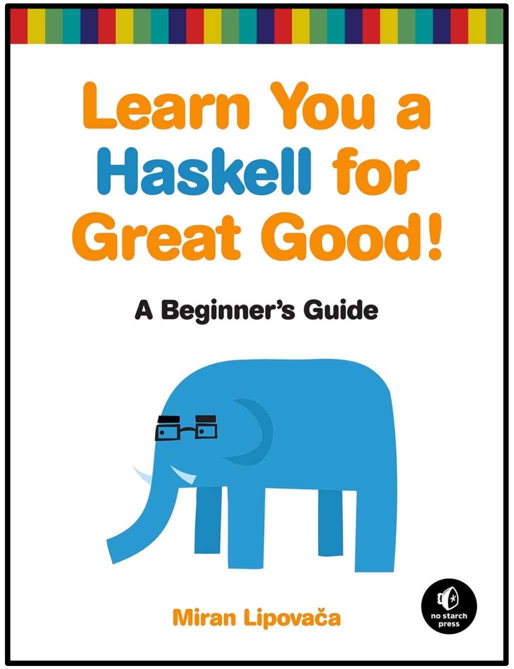
```
ghci> (+) <$> (+3) <*> (*100) $ 5
508
```

Calling <*> with two applicative values results in an applicative value, so if we use it on two functions, we get back a function. So what goes on here? When we do `(+) <$> (+3) <*> (*100)`, we're making a function that will use `+` on the results of `(+3)` and `(*100)` and return that. With `(+) <$> (+3) <*> (*100) $ 5`, `(+3)` and `(*100)` are first applied to 5, resulting in 8 and 500. Then `+` is called with 8 and 500, resulting in 508.

The following code is similar:

```
ghci> (\x y z -> [x,y,z]) <$> (+3) <*> (*2) <*> (/2) $ 5
[8.0,10.0,2.5]
```

We create a function that will call the function `\x y z -> [x,y,z]` with the eventual results from `(+3)`, `(*2)` and `(/2)`. The 5 is fed to each of the three functions, and then `\x y z -> [x,y,z]` is called with those results.




Miran Lipovača

Just like when we looked at Miran's Lipovača's example of using Functor's <$> operator, I think that the use of partially applied functions like (*3) and (+100) could be making the examples on the previous slide slightly harder to understand, by adding some unnecessary complexity. So here is the first example again, but this time using single-argument functions twice and square.

```
λ (+) <$> twice <*> square $ 5
35

λ ((+) <$> twice <*> square) 5
35

λ (((+) . twice) <*> square) 5
35

λ (((+) . twice) 5)(square 5)
35

λ ((+) . twice) 5 (square 5)
35

λ (+) (twice 5) (square 5)
35
```

```
λ inc n = n + 1
λ twice n = n + n
λ square n = n * n

λ :type inc
inc :: Num a => a -> a

λ :type twice
twice :: Num a => a -> a

λ :type square
square :: Num a => a -> a

λ :type (+) <$> twice <*> square
(+) <$> twice <*> square :: Num b => b -> b
```

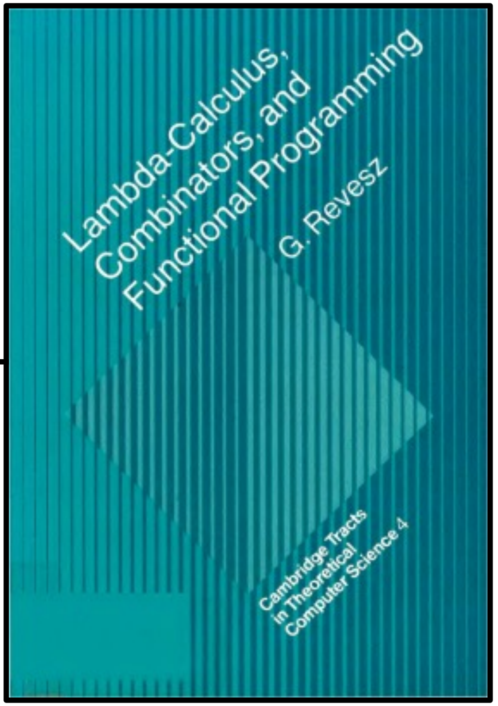
The other example would look like this: list3 <$> inc <*> twice <*> square $ 5. Instead of working through that right now, we are going to first see how the functions <$>, pure and <*> relate to combinatory logic, and then come back to both examples and work through them by viewing the functions as combinators. This will make evaluating the expressions in the examples easier to understand. It will also be quite interesting.

It turns out that the three functions of the **Applicative** instance for **functions**, i.e. **fmap**, **pure** and **<*>**, are known in **combinatory logic** as **combinators**. The first one is called **B** and can be defined in terms of the other two, which are called **K** and **S**, and which are also called **standard combinators**. See the next slide for a table summarising some key facts about the combinators mentioned on this slide.

According to a fundamental theorem of **Schönfinkel** and **Curry**, the entire **$\lambda$-calculus** can be recast in the **theory of combinators**, which has only one basic operation: **application**. **Abstraction** is represented in this theory with the aid of two **distinguished combinators**: S and K, which are called **standard combinators**.

…

The two **standard combinators**, S and K, are sufficient for eliminating all abstractions, i.e. all bound variables from every **$\lambda$-expression**.




A **combinator** is a **$\lambda$-expression** in which there are no occurrences of free variables. For example, the **identity function** $\lambda x.\, x$ is a **combinator** and is usually referred to by the identifier I. Another example is the **fixed-point combinator** … which is defined by $\mathrm{Y} = \lambda h.\, ((\lambda x.\, h(x\ x))(\lambda x.\, h(x\ x)))$.

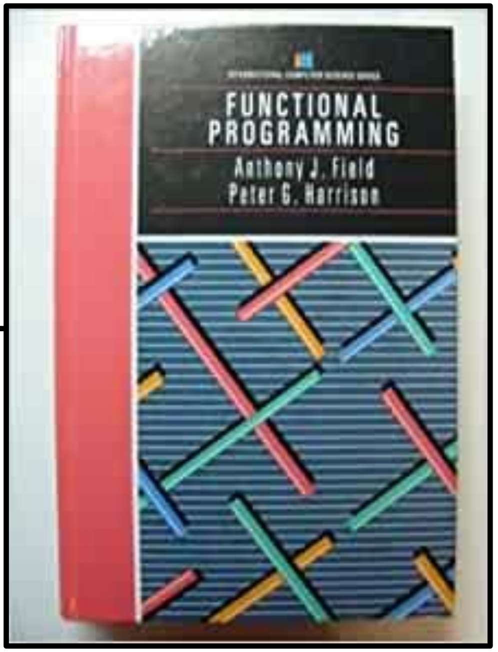
Two further examples are the **cancellator** K given by $\lambda x.\lambda y.\, x$, and the **distributor** S given by $\lambda f.\lambda g.\lambda x.\, f\ x(g\ x)$. (Incidentally, the names for these are K and S rather than C and D because they were invented by Germans.)

Now, as we shall see, any **$\lambda$-expression** E can be converted into an **applicative expression**, i.e. an expression built entirely from function applications, **lambda abstractions** thereby being **absent**. To achieve this we require at least the two **combinators** (functions) S and K to be included in the expression syntax as additional constants. In fact, the **$\lambda$-calculus** and the **combinatory logic** defined on these **combinators** are equivalent …

In our presentation we shall also use the **identity combinator** I, although it should be noted that it can be defined in terms of S and K using the identity I=SKK. …

…

Although the complexity of the **combinatory logic** expressions which use only the S, K and I **combinators** is unacceptably high for use as a viable implementation technique, certain sub-expressions structures have much simpler forms, which are equivalent… Moreover, a much larger number of expressions may be simplified similarly if we introduce two further **primitive combinators** into the fixed set, using corresponding new identities. The new **combinators** in question are called the **compositor**, denoted by B, and the **permutator**, denoted by C, and are defined in the **$\lambda$-calculus** by $\mathrm{B} = \lambda f.\lambda x.\lambda y.\, f(x\ y)$ and $\mathrm{C} = \lambda f.\lambda x.\lambda y.\, f\ y\ x$.



| Name (Curry) | Name (Smullyan) | Definition | Haskell function | Signature | Alternative name and Lambda function | Name (Schönfinkel) | Definition in terms of other combinators |
|---|---|---|---|---|---|---|---|
| S | Starling | $S\ f\ g\ x = f\ x\ (g\ x)$ | `Applicative's (<*>) on functions` | `(a -> b -> c) -> (a -> b) -> a -> c` | Distributor<br>$\lambda x.y.z.x\ z(y\ z)$ | S<br>Verschmelzungsfunktion (amalgamation function) | |
| K | Kestrel | $K\ x\ y = x$ | `const` | `a -> b -> a` | Cancellator<br>$\lambda x.y.x$ | K<br>Konstanzfunktion (constant function) | |
| I | Identity Bird | $I\ x = x$ | `id` | `a -> a` | Idiot<br>$\lambda x.x$ | I<br>Identitätsfunktion (identity function) | SKK |
| B | Bluebird | $B\ f\ g\ x = f(g\ x)$ | `.` | `(b -> c) -> (a -> b) -> a -> c` | Compositor<br>$\lambda x.y.z.x(y\ z)$ | Z<br>Zusammensetzungsfunktion (composition function) | S(KS)K |
| C | Cardinal | $C\ f\ x\ y = f\ y\ x$ | `flip` | `(a -> b -> c) -> b -> a -> c` | Permutator<br>$\lambda x.y.z.x\ z\ y$ | T<br>Vertauschungsfunktion (exchange function) | S(BBS)(KK) |

To Mock a Mockingbird - Raymond Smullyan
Functional Programming - Anthony J. Field, Peter G. Harrison
Lambda Calculus, Combinators and Functional Programming – G. Revesz.
Recursive Programming Techniques– W. H. Burge

https://www.angelfire.com/tx4/cus/combinator/birds.html
https://www.johndcook.com/blog/2014/02/06/schonfinkel-combinators/
http://hackage.haskell.org/package/data-aviary-0.4.0/docs/Data-Aviary-Birds.html

The previous two slides described combinators S, K, I, B and C. The ones that we are interested in are S, K, and B, because they are the following functions of the **Applicative** instance for **functions**: **fmap**, **pure** and **<*>**.

| | | | | |
|---|---|---|---|---|
| Bluebird | $B\ f\ g\ x = f(g\ x)$ | Compositor | B | **fmap**, (<$>) |
| Kestrel | $K\ x\ y = x$ | Cancellator | K | **pure** |
| Starling | $S\ f\ g\ x = f\ x\ (g\ x)$ | Distributor | S | (<*>) |

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

```
class Functor f => Applicative f where
   pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

```
instance Functor ((->) r) where
  fmap = (.)
```

```
(<$>) :: Functor f => (a -> b) -> f a -> f b
(<$>) = fmap
```

```
instance Applicative ((->) r) where
  pure x = (\_ -> x)
  f <*> g = \x -> f x (g x)
```

```
(+) <$> twice <*> square $ 5

(((+) <$> twice) <*> square) 5

(s (b (+) twice) square) 5
   (b (+) twice) 5 (square 5)
      (+)(twice 5) (square 5)
      (+) 10 25
      35
```

```
-- Combinators

-- B: compositor
b f g x = f (g x)

-- K: cancellator
k x y = x

-- S: distributor
s f g x = f x (g x)
```

```
pure (+) <*> twice <*> square $ 5

(((pure (+)) <*> twice) <*> square) 5

(s (s (k (+)) twice) square) 5
   (s (k (+)) twice) 5 (square 5)
      (k (+)) 5 (twice 5) (square 5)
         (+) (twice 5) (square 5)
         (+) 10 25
         35
```

```
list3 <$> inc <*> twice <*> square $ 5

(((list3 <$> inc) <*> twice) <*> square) $ 5

(s (s (b list3 inc) twice) square) 5

(s (b list3 inc) twice) 5 (square 5)

(b list3 inc) 5 (twice 5) (square 5)

list3 (inc 5) (twice 5) (square 5)

list3 6 10 25

[6,10,25]
```

```
list3 x y z = [x,y,z]
```

```
(pure list3 <*> inc <*> twice <*> square) 5

((((pure list3) <*> inc) <*> twice) <*> square) 5

s (s (s (k list3) inc) twice) square 5

(s (s (k list3) inc) twice) 5 (square 5)

(s (k list3) inc) 5 (twice 5) (square 5)

(k list3) 5 (inc 5) (twice 5) (square 5)

list3 (inc 5) (twice 5) (square 5)

list3 6 10 25

[6,10,25]
```

So what does the Scala equivalent of the Haskell implementation of the Applicative instance for functions look like?

Here it is

```haskell
class Functor f => Applicative f where
  pure :: a -> f a
  (<*>) :: f (a -> b) -> f a -> f b
```

```haskell
instance Applicative ((->) r) where
  pure x = (\_ -> x)
  f <*> g = \x -> f x (g x)
```

```haskell
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

```haskell
(<$>) :: (Functor f) => (a -> b) -> f a -> f b
f <$> x = fmap f x
```

```haskell
instance Functor ((->) r) where
  fmap = (.)
```

```scala
trait Functor[F[_]]:
  def map[A,B](f: A => B): F[A] => F[B]
  extension[A,B] (f: A => B)
    def `<$>`(fa: F[A]): F[B] = map(f)(fa)

trait Applicative[F[_]] extends Functor[F]:
  def pure[A](a: => A): F[A]
  def apply[A,B](fab: F[A => B]): F[A] => F[B]
  extension[A,B] (fab: F[A => B])
    def <*> (fa: F[A]): F[B] = apply(fab)(fa)

given functionApplicative[D]: Applicative[[C] =>> D => C] with
  override def pure[A](a: => A): D => A = x => a
  override def map[A,B](f: A => B): (D => A) => (D => B) =
    g => f compose g
  override def apply[A,B](f: D => A => B): (D => A) => (D => B) =
    g => n => f(n)(g(n))
```

And here are some **Scala** tests showing the **Applicative** for **functions** in action.

```scala
val inc: Int => Int = x => x + 1
val twice: Int => Int = x => x + x
val square: Int => Int = x => x * x
def list3[A]: A => A => A => List[A] = x => y => z => List(x,y,z)

assert( (map(map(inc)(twice))(square))(3) == 19)
assert( (inc `<$>` twice `<$>` square)(3) == 19)

assert( (pure(`(+)`) <*> twice <*> square)(5) == 35)
assert( (`(+)` `<$>` twice <*> square)(5) == 35)

assert( (pure(list3) <*> inc <*> twice <*> square)(5) == List(6,10,25) )
assert( (list3 `<$>` inc <*> twice <*> square)(5) == List(6,10,25) )
```

It can be a bit hard to visualise how a **function type** can instantiate the **Applicative** type class, so here is a diagram that helps you do that by contrasting the signatures of **pure**, **fmap**, and (**<*>**)  in two **Applicative** instances, one for **Maybe** and one for **functions**.

@philip_schwarz

```
pure                a -> f a
fmap         (a -> b) -> f a -> f b
(<*>)      f (a -> b) -> f a -> f b
```

```
f a
⟹
Maybe a
```

Maybe Applicative

```
pure                a -> Maybe a
fmap         (a -> b) -> Maybe a -> Maybe b
(<*>)  Maybe (a -> b) -> Maybe a -> Maybe b
```

```
f a
⟹
(d -> a)
∃(d) e.g. Int
```

((->) d) Applicative

```
pure                 a -> (d -> a)
fmap          (a -> b) -> (d -> a) -> (d -> b)
(<*>)   (d -> a -> b) -> (d -> a) -> (d -> b)
```

```
pure       (a:      => A) : F[A]
Map        (f:    A => B) : F[A] => F[B]
<*>      (fab: F[A => B]): F[A] => F[B]
```

```
F[A]
⟹
Option[A]
```

Option Applicative

```
Pure      (a:           => A) : Option[A]
Map       (f:         A => B) : Option[A] => Option[B]
<*>   (fab: Option[A => B]): Option[A] => Option[B]
```

```
F[A]
⟹
D => A
∃(D) e.g. Int
```

([C] =>> D => C) Applicative

```
pure       (a:           => A):  D => A
Map        (f:         A => B): (D => A) => (D => B)
<*>     (fab: D => A => B): (D => A) => (D => B)
```

Now that we are thoroughly familiar with the **Applicative** for **functions**, it's finally time to see, on the next slide, how the **palindrome checker** function, which inspired this slide deck, works.

```haskell
(≡) :: Eq a => a -> a -> Bool
(≡) x y = x == y

reverse :: [a] -> [a]

(<*>) :: Applicative f => f (a -> b) -> f a -> f b
(<*>) f g x = f x (g x)

palindrome :: Eq a => [a] -> Bool
palindrome = (≡) <*> reverse

((->) d) Applicative
(<*>) :: (d -> a -> b) -> (d -> a) -> (d -> b)

d = [Char]
a = [Char]
b = Bool
```

```haskell
-- S, the Distributor
-- combinator, aka Starling
s f g x = f x (g x)
```

```haskell
palindrome "amanaplanacanalpanama"

((≡) <*> reverse) "amanaplanacanalpanama"

(s (≡) reverse) "amanaplanacanalpanama"

((≡) "amanaplanacanalpanama") (reverse "amanaplanacanalpanama")

((≡) "amanaplanacanalpanama") "amanaplanacanalpanama"

"amanaplanacanalpanama" ≡ "amanaplanacanalpanama"

True
```

```haskell
(<*>) :: ([Char] -> [Char] -> Bool) -> ([Char] -> [Char]) -> ([Char] -> Bool)
```

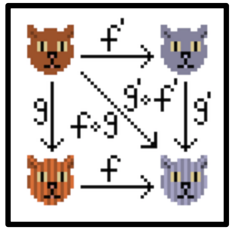
(≡) reverse palindrome

```haskell
λ palindrome "amanaplanacanalpanama”
True
```

```
trait Functor[F[_]]:
  def map[A,B](f: A => B): F[A] => F[B]
  extension[A,B] (f: A => B)
    def `<$>`(fa: F[A]): F[B] = map(f)(fa)

trait Applicative[F[_]] extends Functor[F]:
  def pure[A](a: => A): F[A]
  def apply[A,B](fab: F[A => B]): F[A] => F[B]
  extension[A,B] (fab: F[A => B])
    def <*> (fa: F[A]): F[B] = apply(fab)(fa)

given functionApplicative[D]: Applicative[[C] =>> D => C] with
  override def pure[A](a: => A): D => A = x => a
  override def map[A,B](f: A => B): (D => A) => (D => B) =
    g => f compose g
  override def apply[A,B](f: D => A => B): (D => A) => (D => B) =
    g => n => f(n)(g(n))
```

We have already seen hand rolled **Scala** code for the **Applicative** type class and the **Applicative** instance for **functions**. Here it is again, with some new additional code implementing and testing the **palindrome checker** function. Not how if we use the **Cats** library we only need a few imports to get the new code to work.

```
import cats._
import cats.implicits._
import cats.Applicative
```



**Option 1**

Hand rolled **Applicative** type class and implicit instance for **functions**.

```
def ≣[A] = ((_: A) == (_: A)).curried

def reverse[A]: Seq[A] => Seq[A] = _.reverse

def palindrome[A]: Seq[A] => Boolean =
  ≣ <*> reverse

@main def main =

  assert(   palindrome("amanaplanacanalpanama") )
  assert( ! palindrome("abcabc") )
  assert(   palindrome(List(1,2,3,3,2,1)) )
  assert( ! palindrome(List(1,2,3,1,2,3)) )
```

**Option 2**

Predefined **Cats Applicative** type class and predefined implicit instance for **functions**.

The next slide is the last one and it is just a recap of the **Applicative** instances we have seen in this deck.

```haskell
class Functor f where
fmap :: (a -> b) -> f a -> f b
```

```haskell
class Functor f => Applicative f where
pure :: a -> f a
(<*>) :: f (a -> b) -> f a -> f b
```

```haskell
instance Functor Maybe where
fmap _ Nothing  = Nothing
fmap f (Just a) = Just (f a)
```

```haskell
instance Applicative Maybe where
pure = Just
Nothing <*> _ = Nothing
Just f  <*> m = fmap f m
```

```haskell
instance Functor [] where
fmap = map
```

```haskell
instance Applicative [] where
pure x = [x]
fs <*> xs = [f x | f <- fs, x <- xs]
```

```haskell
map :: (a -> b) -> [a] -> [b]
map _ []     = []
map f (x:xs) = f x : map f xs
```

```haskell
instance Functor IO where
fmap f action = do
  result <- action
  return (f result)
```

```haskell
instance Applicative IO where
pure = return
a <*> b = do
  f <- a
  x <- b
  return (f x)
```

```haskell
instance Functor ((->) r) where
fmap = (.)
```

```haskell
instance Applicative ((->) r) where
pure x = (\_ -> x)
f <*> g = \x -> f x (g x)
```

That’s all. I hope you found that useful.

@philip_schwarz